



DE

TRIANGULORUM RECTILINEORUM PROPRIETATIBUS QUIBUSDAM NONDUM SATIS COGNITIS.

MEMORIAM ANNIVERSARIAM

INAUGURATAE ANTE HOS CCLXXXII. ANNOS

SCHOLAE PROVINCIALIS PORTENSIS

A. D. I. NOVEMBR. MDCCCXXV.

PIE CELEBRANDAM

INDICIT

ET

AD RECITATIONES ET ORATIONES DISCIPULORUM

EX OMNIBUS CLASSIBUS ELECTORUM

AUDIENDAS

INVITAT

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1825.

Saepenumero geometria laudibus celebrata est, quod omnia, quae ab ea tractantur, ita sunt clara atque perspicua, ut nullum omnino dubitationi locum relinquant, sed potius cuique, vel invito, assensum extorqueant, eamque ob caussam humano ingenio excolendo et acuendo optime inserviant. Non plane defuerunt quidem, qui hanc geometriae gloriam imminuere, vel ei eripere conarentur; quum autem hi viri plerumque iis ipsis, quae de hac re disputarunt, se veram hujus disciplinae naturam atque indolem non habuisse satis perspectam, ostenderint, tantum abfuit, ut geometria detrimenti aliquid ab iis ceperit, ut contra melius firmata ac munita abiisse sit censenda. Eadem vero alteram, eamque, ut nobis videtur, non minorem laudem sibi vindicat ex ea virtute, qua ita comparata est, ut singuli fere dies vera ac laeta incrementa [1]) ei ferant, dum indefesso quodam studio non solum res novas sibi tractandas sumit, quo multis aliis disciplinis opem fert saluberrimam, sed etiam in iis, quas jam cognitas habet, nova ratione tractandis eoque magis cum explanandis et poliendis, tum amplificandis versatur. Geometria agrum quasi conspiciendum nobis praebet ingentem et vere immensum, cujus pars alia jam ab antiquis gentibus, maxime autem a Graecis ita quidem est culta, ut generi humano per multa secula fructus uberrimos tulerit, cui vero etiamnum geometrarum opera, quae fructuosiorem reddat, nunquam deest; alia deinde ab ineunte seculo proxime elapso coli coepit, in qua magis magisque colenda ejusque finibus ulterius proferendis haud paucos viros, quorum aliquot divino paene ingenio praeditos esse putes, omnem operam ponere cernimus; alia denique posteritatis diligentiam et assiduitatem exspectat. Ad hanc nostram sententiam confirmandam non ultimum locum obtinent ea, quae nuper duo clarissimi viri, Augustus Leopoldus Crelle [2]) et Carolus Guilielmus

[1]) „Die Eroberungen der Mathematik wachsen täglich, wiewohl ohne Geräusch." *Herbart* über die Möglichkeit, Mathematik auf Psychologie anzuwenden. Königsberg 1822. p. 47.

[2]) Über einige Eigenschaften des ebenen geradlinigen Dreiecks etc. Berlin 1816.

Feuerbach [*]) in lucem ediderunt. Qui quidem viri, quum et plures novas triangulorum rectilineorum proprietates tradiderint, et alias, jam antea cognitas, nova et magis expedita tractarint ratione, luculenter docuerunt, quam egregie errarent ii, qui vel hanc geometriae partem, in qua de simplicissimo figurarum rectilinearum genere agitur, plane absolutam omnisque amplificationis expertem esse putarent. Ipsa autem ea, quae hi viri protulerunt, partim augeri, partim alia via demonstrari possunt. Itaque hac scribendi occasione oblata, mihi, sive lectores, quibus potissimum haec commentatio scribitur, respicerem, sive summi geometrae [*]) de ejusmodi re sententiam in memoriam mihi revocarem, non est visum alienum, quas inveni novas et propositionum, a cl. Crellio allatarum, demonstrationes, et similes iis, quas cl. Feuerbachius docuit, triangulorum proprietates, publici juris nunc facere.

Sectio commentationis prima.

De lineis transversis, in uno puncto invicem sese transeuntibus.

§. 1.

Quae lineae AD, BE, CF (Fig. 1 et 2) in triangulo quopiam ABC, cujus latera literis a, b, c, angulos vero A, B, C, more geometrarum consueto, significari volumus, ab angulorum verticibus ita ducuntur, ut lateribus adversis vel ipsis, vel productis occurrant, has *lineas transversas* adpellemus; praeterea autem brevitatis caussa esse ponamus:

$$BD=a' \quad CD=a'' \quad AD=d \quad AK=d' \quad IE=e''$$
$$CE=b' \quad AE=b'' \quad BE=e \quad HD=d'' \quad CI=f'$$
$$AF=c' \quad BF=c'' \quad CF=f \quad BH=e' \quad FK=f''$$

$$KH=i \quad <BAD=A' \quad <CAD=A''$$
$$IH=k \quad <CBE=B' \quad <ABE=B''$$
$$KI=h \quad <ACF=C' \quad <BCF=C''$$

Quibus quidem positis, ex notissima triangulorum proprietate esse, patet:

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*) Eigenschaften einiger merkwürdigen Punkte des geradlinigen Dreiecks etc. Nürnberg 1822.

*) Laplace exposition du systeme du monde 4me edit. Par. 1813. pag. 403. „tout se tient dans la chaine immense des verités de la Geometrie pure; et souvent une seule observation a suffi pour feconder les plus steriles en apparence etc.

$$\begin{array}{lllll} a' : d & :: \sin A' & : \sin B \\ d : a'' & :: \sin C & : \sin A'' \\ b' : e & :: \sin B' & : \sin C \\ e : b'' & :: \sin A & : \sin B'' \\ c' : f & :: \sin C' & : \sin A \\ f : c'' & :: \sin B & : \sin C'' \end{array}$$

$$a'.b'.c' : a''b''c'' :: \sin A'.\sin B'.\sin C' : \sin A'' \sin B''.\sin C''$$

h. e. parallelepipeda recta, a ternis laterum fragmentis non contiguis facta, eandem inter se rationem habent, quam multiplicatis angulorum illis oppositorum sinibus quae efficiuntur quantitates.

Cor. Simili plane ratione, proportionem hanc

$$d'.e'.f' : (d'+i)(e'+k)(f'+h) :: \sin A'.\sin B'.\sin C' : \sin A''.\sin B''.\sin C''$$

veram esse, probatur; unde facile colligimus, esse:

$$a'.b'.c' : a''.b''.c'' :: d'.e'.f' : (d'+i)(e'+k)(f'+h)$$

§. 2.

Trianguli GHI, a lineis transversis conditi, magnitudinem non semper sibi constare, sed potius cum transversis ipsis mutari, per se satis adparet; nec minus, eam plane evanescere tum, quum puncta G, H, I unum idemque fiant, vel lineae transversae eodem invicem secentur puncto, facile perspicitur. Tunc vero sit

$$g = h = i = o$$

necesse est, quare quas modo tradidimus proportiones, mutari videmus in has:

$$d'.e'.f' : d'.e'.f' :: \sin A'.\sin B' \sin C' : \sin A''.\sin B''.\sin C''$$

$$d'.e'.f' : d'.e'.f' :: a'.b'.c' : a''.b''.c''$$

unde

$$\sin A'.\sin B'.\sin C' = \sin A''.\sin B''.\sin C''$$

$$\text{et } a'.b'.c' = a''.b''.c''$$

h. e. si lineae transversae ita ducantur, ut unum punctum transeant, non solum parallelepipeda recta, a ternis laterum partibus non contiguis facta, sed etiam quae ternorum, qui illis adversi sunt, angulorum, neque vicinorum, neque eidem trianguli lateri adjacentium, sinibus multiplicatis gignuntur quantitates, aequantur *).

*) Primus hoc theorema demonstravit Johannes Bernoulli v. Opp. Tom. IV. p. 22; deinde alias demonstrationes protulerunt: Carnot v. Geometrie der Stellung übers. von Schumacher Tom. II. p. 329; Crelle in libro laudato p. 7. et alii.

§. 3.

Quae lineae transversae in triangulo ita ducuntur, ut sit (Fig. 3.)

$$a'.b'.c' = a''.b''.c''$$

in eodem punctó has invicem secari necesse est. Fingamus enim, non esse, quae diximus, vera; sequetur, ducta recta AGH, esse

$$BH.b'.c' = CH.b''.c'',$$

inde, quum ex hyp. sit $a'.b'.c' = a''.b''.c''$

$$a' : BH :: a'' : CH$$
$$a'+a'' : BH+CH :: a'' : CH$$

ergo $a'' = CH$ esse oportet; quod vero plane absurdum; ergo fieri nequit, quin, si $a'.b'.c' = a''.b''.c''$, punctum sit quoddam G, quod transversae una transcant.

Cor. Non minus lineas transversas, ita ductas, ut sit

$$\sin A'.\sin B'.\sin C' = \sin A''.\sin B''.\sin C''$$

eodem puncto secari, facile e propositione nostra et illa, quam (§. 2.) adtulimus, colligitur.

Alia hujus theorematis demonstratio hoc modo confici potest.

Si unquam fieri posset, ut transversae AD, BE, CF (Fig. 3), licet ita ductae, ut esset

$$\sin A'.\sin B'.\sin C' = \sin A''.\sin B''.\sin C''$$

idem punctum non transirent, omnino sequeretur, ducta AGH angulisque BAH et CAH per literas A''', A^{iv} denotatis, esse

$$\sin A'''.\sin B'.\sin C' = \sin A^{\text{iv}}.\sin B''.\sin C'';$$

est autem ex hyp.

$$\sin A'.\sin B'.\sin C' = \sin A''.\sin B''.\sin C'';$$

ergo $\sin A' : \sin A'' :: \sin A''' : \sin A^{\text{iv}}$

$$\sin A'+\sin A'' : \sin A'''+\sin A^{\text{iv}} :: \sin A'-\sin A'' : \sin A'''-\sin A^{\text{iv}}$$

$$\sin\tfrac{1}{2}A.\cos\tfrac{1}{2}(A'-A'') : \sin\tfrac{1}{2}A.\cos\tfrac{1}{2}(A'''-A^{\text{iv}}) :: \sin\tfrac{1}{2}(A'-A'').\cos\tfrac{1}{2}A : \sin\tfrac{1}{2}(A'''-A^{\text{iv}}).\cos\tfrac{1}{2}A$$

$$1 : 1 :: \operatorname{tg}\tfrac{1}{2}(A'-A'') : \operatorname{tg}\tfrac{1}{2}(A'''-A^{\text{iv}});$$

ergo

$$A'+A^{\text{iv}} = A''+A'''$$

quod quidem, quum

$$A' < A'''$$
$$A^{\text{iv}} < A''$$

esse pateat, fieri plane nequit; ergo enunciatio nostra vera sit necesse est.

§. 4.

His constitutis plura facili negotio intelligi possunt [6]). Nimirum

1) in eodem puncto lineae transversae sese transeant necesse est eae, quae
   a) aut latera, aut angulos trianguli in partes aequales dividunt;
   b) lateribus adversis ad angulos rectos occurrunt [7]); est enim tunc $A' = C''$, $B' = A''$, $C' = B''$;
   c) latera iisdem punctis secant, quibus circulus, triangulo inscriptus, illa tangit; nunc enim

$$a' = c'', \quad b' = a'', \quad c' = b''$$

esse patet.

2) Quando trianguli cujuspiam ABC duo latera AB, AC (Fig. 4.) in duas partes secantur, quae, si singulam singulae compares, eadem, qua latera ipsa, inter se sunt ratione, ductis lineis transversis, tertium latus BC, a recta AGD in duas aequas portiones dividi contendimus, et contra. Est enim non solum

$$\text{et} \begin{array}{l} c' : b'' :: c : b \\ b' : c'' :: b : c \end{array}$$

$$\text{unde} \quad b'.c' = b''.c'', \quad \text{sed etiam}$$

$$a'.b'.c' = a''.b''.c'',$$

ex quo concludere licet, esse

$$a' = a'' = \tfrac{1}{2}\, a \,.$$

Contra quae linea latus a bifariam secat, eandem punctum G, quo BF, CF invicem secantur, transire, jam ex eo adparet, quod a dato puncto ad datam lineam una tantummodo duci potest recta, quae alteram in duas partes aequales dividit.

3) Quodsi in utroque latere b, c numerum partium infinitum, AE, AF, AH, AI, AL, AM etc. (Fig. 5.) ita sumseris, ut sit:

$$b : c :: AE : AF :: AH : AI :: AL : AM \text{ etc.}$$

lineasque EI, FH, HM, IL, LB, MC etc. duxeris, puncta G, K, N etc., quibus binae hae rectae invicem secantur, omnia lineam transversam eam, quae tertium

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[6]) cf. Crelle in l. l. p. 10. sqq.

[7]) Elegantissimam hujus theorematis demonstrationem, methodo synthetica perpetratam, quae nondum satis a geometris cognita esse videtur, tradidit Gaufs; v. Carnot Geometrie der Stellung T. II. p. 565.

latus a aequaliter dividit, transire necesse est. Ducantur enim adjutrices *HI*, *LM*, quas, cum sint inter se et lateri a parallelae, per eandem transversam, quae a bifariam secat, in duas aequas portiones dividi patet. Itaque si $BD = DC$, transversa *AD*, ex iis, quae antea demonstravimus, punctum *N* transeat necesse est, nec minus punctum *K*, quoniam et rectam *ML* aequaliter dividit, et sic de ceteris [*]).

§. 5.

In omni autem triangulo praeter eas, quas memoravimus, duci posse innumerabiles transversarum, in eodem puncto sese transeuntium, terniones, per se facile intelligitur; e quarum permagno numero tantum singulas quasdam nunc eligere et paulo accurratius tractare placet.

Sed ne enunciationes nostrae nimia verborum copia laborent, eam transversarum ternionem adpellabimus

primam, qua latera trianguli in aequas portiones dividuntur;
secundam, quae angulos bifariam secat;
tertiam, cujus lineae sunt perpendicula, in trianguli latera demissa;
quartam, quae angulos ita secat, ut sit

$$A' = B' = C';$$

quintam, qua anguli A'', B'', C'' inter se aequales redduntur;
sextam, quae ita comparata est, ut sit

$$a' = c'', \ b' = a'', \ c = b'';$$

septimam denique, quae latera ita secat, ut sit

$$a' = b'', \ b' = c'', \ c' = a''.$$

Signa a', b', c', etc. servabunt quidem eam, quam supra iis tribuimus, significationem, sed superposito numero eo, qui ad quotam ternionem sint referenda indicet.

§. 6.

Quod attinet ad primam ternionem, plura quidem omnino digna, quae memorentur, afferri possent, v. c.

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*) Similem, sed non tam late patentem, propositionem tradidit Puissant in libro: Recueil de diverses propositions de geometrie Par. 1801. p. 50.

1) punctum, quo transversae hae secantur, illud ipsum est, quod centrum gravitatis vocant,

2) $\overset{(1)}{d'} = \overset{(1)}{2d''}$; $\overset{(1)}{e'} = \overset{(1)}{2.e''}$; $\overset{(1)}{f'} = \overset{(1)}{2f''}$

3) $d^2 + \overset{(1)}{e^2} + f^2 = \frac{3}{4}\,(a^2+b^2+c^2)$

4) $d.d' + \overset{(1)}{e.e'} + f.f' = \frac{1}{2}\,(a^2+b^2+c^2)$

5) $d.d'' + \overset{(1)}{e.e''} + f.f'' = \frac{1}{4}\,(a^2+b^2+c^2) = a'^2 + \overset{(1)}{b'^2} + c'^2 = \overline{DE}^2 + \overset{(1)}{\overline{DF}^2} + \overline{EF}^2$

6) quando circulus arbitrariae magnitudinis e puncto G, tanquam centro, describitur, in cujus circuitu P, Q (Fig. .6) duo quaelibet sunt puncta, nunquam non esse potest:

$$\overline{PA}^2 + \overline{PB}^2 + \overline{PC}^2 = \overline{QA}^2 + \overline{QB}^2 + \overline{QC}^2$$

et sic plura alia, quibus vero omnibus, quum sint satis nota, immorari non opus est.

§. 7.

Jam igitur ad secundam, quae angulos aequaliter dividit, ternionem transeamus. Primum autem ex elementis satis constat, esse

$$\overset{(2)}{a'}.b = \overset{(2)}{a''}.c \ ; \ \overset{(2)}{b'}.c = \overset{(2)}{b''}a \ ; \ \overset{(2)}{c'}.a = \overset{(2)}{c''}.b$$

inde

$$a'.(\overset{(2)}{b'}+b'') + b'.(\overset{(2)}{c'}+c'') + c'(\overset{(2)}{a'}+a'') = a''(\overset{(2)}{c'}+c'') + b''(\overset{(2)}{a'}+a'') + c''(\overset{(2)}{b'}+b'')$$

$$a'.b' + b'.c' + \overset{(2)}{a'.c'} + a'.b'' + b'.c'' + a''c' = a''.c'' + a''.b'' + \overset{(2)}{b''.c''} + a''.c' + a'.b'' + b'.c''$$

$$a'b' + \overset{(2)}{a'.c'} + b'.c' = a''.b'' + \overset{(2)}{a''.c''} + b''.c''$$

h. e. tria rectangula, a binis laterum partibus non contiguis facta et in unam summam collecta, trium ceterorum rectangulorum, eadem ratione ortorum, summam aequant.

§. 8.

In omni triangulo, si quatuor circulorum, ita descriptorum, ut quisque latera ejus contingat, radii per literas r, r', r'', r''' exprimantur, esse

$$ab+ac+bc = r.r'+r.r''+r.r'''+r'.r''+r'.r'''+r''.r''',$$

quum cl. Feuerbachius demonstraverit [9]), ex theoremate praecedenti facile sequitur, triangula ea, quorum alteri sunt latera $a', \overset{(2)}{b'}, c'$, alteri vero $a'', \overset{(2)}{b''}, c''$ omnino ita esse comparata, ut rectangula, a binis circulorum, latera alterius trianguli tangentium, radiis effecta, si in unam summam cogantur, ejusmodi in altero triangulo rectangulorum summae sint aequalia.

§. 9.

In geometriae elementis esse probatur:

$$\overset{(1)}{d^2} = bc - \overset{(1)}{a'.a''} \; ; \; \overset{(1)}{e^2} = ac - \overset{(1)}{b'.b''} \; ; \; f^2 = ab - \overset{(1)}{c'.c''}$$

ergo

$$d^2 + e^2 + f^2 = ab+ac+bc-(a'.a''+b'.b''+c'.c'')$$

h. e. quadratorum, super transversis constructorum, summa aream aequat eam, qua rectangula, a binis lateribus condita, et in unam summam collecta, eorum, quae a cujusque fiunt fragmentis, summam excedit.

Cor. Facile hinc

$$d^2+e^2+f^2+a'.a''+b'.b''+c'.c'' = r.r'+r.r''+r.r'''+r.'r''+r'.r'''+r''.r'''$$

esse colligitur.

§. 10.

Nunquam esse nequit:

$$b : c :: \overset{(1)}{a''} : a'$$
$$b+c : a :: c : a'$$
$$(b+c)^2 : a^2 :: c^2 : a'^2$$
$$(b+c)^2-a^2 : c^2-a'^2 :: (b+c)^2 : c^2$$
$$(a+b+c)\,(-a+b+c) : c^2-a'^2 :: (b+c)^2 : c^2$$

praeterea autem quum sit:

9) in l. l. §. 6.

$$\overset{(2)}{d}{}^2 = b.c - \overset{(2)}{a'}.a'' = bc - \overset{(2)}{a'}{}^2 . \frac{b}{c} = \frac{b}{c}\,(c^2 - \overset{(2)}{a'}{}^2),$$

unde

$$\overset{(2)}{d}{}^2 : b :: c^2 - \overset{(2)}{a'} \quad : c$$
$$\overset{(2)}{d}{}^2 : b.c :: c^2 - \overset{(2)}{a'}{}^2 : c^2\ ;$$

hinc nancis· · ar proportionem hanc:

$$\overset{(2)}{d}{}^2 : bc :: (a+b+c)\ (-a+b+c) : (b+c)^2\ ;$$

similique ratione

$$\overset{(2)}{e}{}^2 : ac :: (a+b+c)\ (a-b+c) : (a+c)^2$$
$$\overset{(2)}{f}{}^2 : ab :: (a+b+c)\ (a+b-c) : (a+b)^2\ ;.$$

ergo

$$d^2.\overset{(2)}{e}{}^2.f^2 : a^2.b^2.c^2 :: (a+b+c)^2(-a+b+c)(a-b+c)(a+b-c) : (a+b)^2(a+c)^2.(b+c)^2$$
$$:: (a+b+c)^2.\ 16\triangle^2 : (a+b)^2\,(a+c)^2\,(b+c)^2$$

eamque ob caussam

$$d.e.f : a.b.c :: (a+b+c)\ 4\ \triangle : (a+b)\ (a+c)\ (b+c)$$

h. e. recta parallelepipeda ea, quorum alterum a transversis, alterum a trianguli lateribus efficitur, eadem inter se sunt ratione, qua illud prisma, cujus basis trianguli aream quater complectitur, altitudo vero ejusdem perimetrum aequat, est ad parallelepipedum, a binorum laterum summis factum.

§. 11.

Demissis a puncto G in trianguli latera perpendiculis GL, GM, GN, (Fig. 7.), esse

$$\overset{(2)}{d'}{}^2 = \overline{AM}^2 + \overline{GM}^2\ ,$$

videmus; quoniam autem

$$AM = \tfrac{1}{2}\,(-a+b+c)\ ,\ \text{et}\ GM = \frac{2\triangle}{a+b+c}$$

esse constat, prodeat nobis necesse est haec aequatio:

$$\overset{(')}{d'^2} = \tfrac{1}{4}(-a+b+c)^2 + \frac{4\,\triangle^2}{(a+b+c)^2}$$

$$\overset{(')}{d'^2}\,(a+b+c)^2 = \tfrac{1}{4}\,(a+b+c)^2\,(-a+b+c)^2 + 4\,\triangle^2$$
$$= \tfrac{1}{4}\,(a+b+c)\,(-a+b+c)\,[(a+b+c)(-a+b+c)+(a-b+c)(a+b-c)]$$
$$= (a+b+c)\,(-a+b+c)\,bc\,;$$

proinde

$$\overset{(')}{d'^2} : bc :: -a+b+c : a+b+c,$$

eademque via nanciscimur:

$$\overset{(')}{e'^2} : ac :: a-b+c : a+b+c$$
$$\overset{(')}{f'^2} : ab :: a+b-c : a+b+c\,;$$

ergo

$$d'^2.\overset{(')}{e'^2}.f'^2 : a^2.b^2.c^2 :: (-a+b+c)\,(a-b+c)\,(a+b-c) : (a+b+c)^3$$
$$:: 8\,r\,\triangle\,[10] : (a+b+c)^3$$
$$:: 4\,r^2 \quad : (a+b+c)^2$$
$$d'.\overset{(')}{e'}.f' \;:\; a.b.c \;::\; 2\,r \;:\; a+b+c$$

h. e. parallelepipeda recta, quorum alterum a partibus transversarum superioribus, alterum fit a trianguli lateribus, eandem inter se habent rationem, quam circuli interioris diametrus ad trianguli perimetrum.

§. 12.

Ex iis, quae modo (§§. 10. et 11.) demonstravimus, est

$$d.\overset{(')}{e}.f :\; a.\,b.\,c :: (a+b+c)\;4\,\triangle : (a+b)\,(a+c)\,(b+c)$$
$$a.\,b.\,c \;:\; d'.\overset{(')}{e'}.f' :: a+b+c \;:\; 2\,r\,;$$

ergo

$$d.\overset{(')}{e}.f \;:\; d'.\overset{(')}{e'}.f' :: (a+b+c)^2\;4\,\triangle : 2\,r\;(a+b)\,(a+c)\,(b+c)$$
$$:: (a+b+c)^2\;4\,\triangle : \frac{4\,\triangle}{a+b+c}\,(a+b)\,(a+c)\,(b+c)$$
$$:: (a+b+c)^3 : (a+b)\,(a+c)\,(b+c)$$

[10]) Litera r significat radium circuli, triangulo inscripti, quem in sequentibus circulum interiorem adpellabimus.

h. e. qua ratione parallelepipedum, a transversis ipsis factum, est ad illud, quod a partibus earum superioribus oritur, eàdem est cubus, cujus latus trianguli perimetrum aequat, ad parallelepipedum a binorum laterum summis effectum.

§. 13.

Facili sane negotio intelligitur esse:

$$< AGE = \tfrac{1}{2}(A+B), \text{ et } < AGM = \tfrac{1}{2}(B+C); \quad \text{(Fig. 7.)}$$
$$\text{ergo} \quad < EGM = \tfrac{1}{2}(A-C),$$

similique ratione esse oportet

$$< FGN = \tfrac{1}{2}(A-B)$$
$$< DGL = \tfrac{1}{2}(B-C),$$

quare sit

$$\overset{(2)}{d'}.\ \sin \tfrac{1}{2} A = \overset{(2)}{e''}.\ \cos \tfrac{1}{2}(A-C)$$
$$\overset{(2)}{e'}\ \sin \tfrac{1}{2} B = \overset{(2)}{f''}.\ \cos \tfrac{1}{2}(A-B)$$
$$\overset{(2)}{f'}.\ \sin \tfrac{1}{2} C = \overset{(2)}{d''}.\ \cos \tfrac{1}{2}(B-C)$$

necesse est; unde

$$d'.\overset{(2)}{e'}.f' : d''.\overset{(2)}{e''}.f'' :: \cos \tfrac{1}{2}(A-B).\cos \tfrac{1}{2}(A-C).\cos \tfrac{1}{2}(B-C) : \sin \tfrac{1}{2} A.\sin \tfrac{1}{2} B.\sin \tfrac{1}{2} C.$$

Quum autem in omni trangulo esse constet

$$a+b : c :: \cos \tfrac{1}{2}(A-B) : \sin \tfrac{1}{2} C$$
$$a+c : b :: \cos \tfrac{1}{2}(A-C) : \sin \tfrac{1}{2} B$$
$$b+c : a :: \cos \tfrac{1}{2}(B-C) : \sin \tfrac{1}{2} A,$$

hanc proportionem locum habere videmus:

$$d'.\overset{(2)}{e'}.f' : d''.\overset{(2)}{e''}.f'' :: (a+b)(a+c)(b+c) : a\,b\,c$$

h. e. recta parallelepipeda ea, quorum alterum a superioribus transversarum partibus, alterum vero ab inferioribus factum est, eadem inter se sunt ratione, qua duo illa, quorum unum a binorum laterum summis, alterum ab ipsis lateribus oritur.

Cor. 1. Itaque in triangulo quovis aequilatero est

$$\overset{(2)}{d'^3} : \overset{(2)}{d''^3} :: 8a^3 : a^3,$$

unde $$\overset{(2)}{d'} : \overset{(2)}{d''} :: 2 : 1$$

h. e. pars cujusque transversae superior inferioris duplum aequat; quod quidem ex elementis satis notum.

Cor. 2. Anguli EGM, FGN, DGL semper ita sunt comparati, ut unus (scilicet is, qui dimidiam inter maximum minimumque angulum differentiam aequat) ceterorum summae aequalis sit. Est enim:

$$< EGM = \tfrac{1}{2}(A-C) = \tfrac{1}{2}(A-B) + \tfrac{1}{2}(B-C) = < FGN + < DGL.$$

Itaque in triangulo aequicruro, quum unus horum angulorum evanescat, duo ceteri inter sint aequales necesse est.

§. 14.

In triangulis ADB, BEC, CFA esse patet:

$$\overset{(2)}{a'} : c :: \overset{(2)}{d''} : d'$$
$$b' : a :: e'' : e'$$
$$c' : b :: f'' : f';$$

unde

$$a'.\, b'.\, c' : a.\, b.\, c :: d''.\, e''.\, f'' : d'.\, e'.\, f'$$

et, quoniam supra

$$d''.\, e''.\, f'' : d'.\, e',\, f' :: a.\, b.\, c : (a+b)\,(a+c)\,(b+c)$$

esse vidimus,

$$a'.\, b'.\, c' : a.\, b.\, c :: a.\, b.\, c : (a+b)\,(a+c)\,(b+c)$$
$$a''.\, b''.\, c'' : a.\, b.\, c :: a.\, b.\, c : (a+b)\,(a+c)\,(b+c)$$

h. e. parallelepipedum rectum, a trianguli lateribus factum, solidum est geometricae proportionis medium inter parallelepipeda ea, quorum unum a tribus laterum fragmentis non contiguis, alterum a binorum laterum summis ortum est.

§. 15.

In iisdem triangulis nunquam esse nequit:

$$\overset{(2)}{d'^2} = b.\, c' - \overset{(2)}{f'}.\, f''$$
$$e'^2 = c.\, a' - d'.\, d''$$
$$f'^2 = a.\, b' - e'.\, e'',$$

unde

$$d'^2 + \overset{(2)}{e'^2} + f'^2 = a\,b' + \overset{(2)}{b}.\,c' + c.\,a' - d'.\,d'' - \overset{(2)}{e'}.\,e'' - f'.\,f''$$

et hanc ob caussam

$$d.\,d' + \overset{(2)}{e}.\,e' + f.\,f' = a\,b' + \overset{(2)}{b}.\,c' + c\,a'$$

h. e. tria rectangula, a singulis transversis earumque fragmentis superioribus facta et in unam summam coacta, rectangulorum a singulo quoque latere et fragmento posterioris **) vicino ortorum summam aequant.

Cor. Facile inde colligitur, esse omnino:

$$d.\,d'' + e.\,e'' + f.\,f'' = (a - b')\,b'' + (b - c')\,c'' + (c - a')\,a''$$

§. 16.

Duarum, quas modo tradidimus, aequationum altera alteri demta prodit haec:

$$d.\,(d' - d'') + e.\,(\overset{(2)}{e'} - e'') + f\,(f' - f'') = a\,b' + b\,c' + c\,a' + (\overset{(2)}{b'} - a)\,b'' + (c' - b)\,c'' + (a' - c)\,a''$$

$$(d' + d'')\,(d' - d'') + (e' + e'')\,(e' - e'') + (f' + f'')\,(f' - f'') =$$

$$(a' + a'')\,b' + (b' + b'')\,c' + (c' + c'')\,a' + (b' - a' - a'')\,b'' + (c' - b' - b'')\,c'' + (a' - c' - c'')\,a'';$$

unde

$$d'^2 + \overset{(2)}{e'^2} + f'^2 - (d''^2 + \overset{(2)}{e''^2} + f''^2) = a''\,(a' + b') + b''\,(b' + c') + c''\,(c' + a') - a'\,b'' - b'\,c'' - c'\,a''$$

$$= (a' + b' + c')\,(a'' + b'' + c'') - 2\,(a'\,b'' + b'\,c'' + c'\,a'')$$

h. e. quantum summa quadratorum, a partibus transversarum superioribus factorum, ab eorum summa differt, quae super inferioribus illarum fragmentis constructa sunt, tantum id rectangulum, quod a ternis laterum partibus non contiguis, in unam summam collectis, efficitur, a dupla illorum summa, quae a binis laterum fragmentis, ex adverso sibi constitutis, oriuntur.

§. 17.

Junctis inter se per rectas transversarum finibus, D, E, F, demissisque ab iisdem punctis in transversas perpendiculis DS, FR, ET, (Fig. 7.) haec aequatio locum habeat necesse est:

**) primum trianguli latus dicimus illud, quod litera a significari volumus, secundum b, tertium denique c; itaque b est latus sequens vel posterius, si latus a respicias, antecedens vero vel prius, si latus c species, etc.

$$\triangle DEF = \triangle DGF + \triangle DGE + \triangle EGF$$
$$= \tfrac{1}{2} [e''. DS + d''. FR + f''. ET]$$
$$= \tfrac{1}{2} r [DS. \sec \tfrac{1}{2}(A-C) + FR. \sec \tfrac{1}{2}(B-C) + ET. \sec \tfrac{1}{2}(A-B)]$$
$$= \tfrac{1}{2} r^2 [\sec \tfrac{1}{2}(A-C). \sec \tfrac{1}{2}(B-C) \sin \tfrac{1}{2}(A+B) + \sec \tfrac{1}{2}(A-B). \sec \tfrac{1}{2}(B-C) \sin \tfrac{1}{2}(A+C) + \sec \tfrac{1}{2}(A-B). \sec \tfrac{1}{2}(A-C) \sin \tfrac{1}{2}(B+C)],$$

unde

$$\triangle DEF = \frac{r^2}{2 \cos \tfrac{1}{2}(A-B). \cos \tfrac{1}{2}(A-C). \cos \tfrac{1}{2}(B-C)} \Big[\sin \tfrac{1}{2}(A+C). \cos \tfrac{1}{2}(A-C) + \sin \tfrac{1}{2}(B+C). \cos \tfrac{1}{2}(B-C) + \sin \tfrac{1}{2}(A+B). \cos \tfrac{1}{2}(A-B)\Big]$$

et, quum esse constet

$$\sin \tfrac{1}{2}(A+C). \cos \tfrac{1}{2}(A-C) = \tfrac{1}{2}(\sin A + \sin C) \text{ etc.,}$$

$$\triangle DEF = \tfrac{1}{4} r^2 \frac{\sin A + \sin B + \sin C}{\cos \tfrac{1}{2}(A-B). \cos \tfrac{1}{2}(A-C). \cos \tfrac{1}{2}(B-C)},$$

unde, quoniam

$$\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A. \cos \tfrac{1}{2} B. \cos \tfrac{1}{2} C$$

esse goniometria docet,

$$\triangle DEF = \frac{2r^2 \cos \tfrac{1}{2} A. \cos \tfrac{1}{2} B. \cos \tfrac{1}{2} C}{\cos \tfrac{1}{2}(A-B). \cos \tfrac{1}{2}(A-C). \cos \tfrac{1}{2}(B-C)}.$$

§. 18.

Unusquisque harum rerum non plane imperitus, e goniometria notissimum habet:

$$\cos \tfrac{1}{2}(A-B) = \cos \tfrac{1}{2} A. \cos \tfrac{1}{2} B. + \sin \tfrac{1}{2} A. \sin \tfrac{1}{2} B.$$
$$\cos \tfrac{1}{2}(A-C) = \cos \tfrac{1}{2} A. \cos \tfrac{1}{2} C + \sin \tfrac{1}{2} A. \sin \tfrac{1}{2} C. \text{ etc.,}$$

quare

$$\cos \tfrac{1}{2}(A-B). \cos \tfrac{1}{2}(A-C). \cos \tfrac{1}{2}(B-C) =$$
$$(\cos \tfrac{1}{2} A. \cos \tfrac{1}{2} B + \sin \tfrac{1}{2} A. \sin \tfrac{1}{2} B). (\cos \tfrac{1}{2} A. \cos \tfrac{1}{2} C + \sin \tfrac{1}{2} A. \sin \tfrac{1}{2} C). (\cos \tfrac{1}{2} B. \cos \tfrac{1}{2} C + \sin \tfrac{1}{2} B. \sin \tfrac{1}{2} C)$$
$$= \cos^2 \tfrac{1}{2} A. \cos^2 \tfrac{1}{2} B. \cos^2 \tfrac{1}{2} C + \tfrac{1}{4} (\sin A. \sin B + \sin A. \sin C. + \sin B. \sin C.) + \sin^2 \tfrac{1}{2} A. \sin^2 \tfrac{1}{2} B. \sin^2 \tfrac{1}{2} C.$$
$$= 1 - \cos^2 \tfrac{1}{2} A - \cos^2 \tfrac{1}{2} B - \cos^2 \tfrac{1}{2} C + \cos^2 \tfrac{1}{2} A. \cos^2 \tfrac{1}{2} B + \cos^2 \tfrac{1}{2} A. \cos^2 \tfrac{1}{2} C + + \cos^2 \tfrac{1}{2} B. \cos^2 \tfrac{1}{2} C + \tfrac{1}{4} [\sin A. \sin B + \sin A. \sin C + \sin B. \sin C.],$$

ergo

$$\triangle DEF = \frac{2r^2 \cos \tfrac{1}{2} A. \cos \tfrac{1}{2} B. \cos \tfrac{1}{2} C.}{1 - \cos^2 \tfrac{1}{2} A - \cos^2 \tfrac{1}{2} B - \cos^2 \tfrac{1}{2} C + \cos^2 \tfrac{1}{2} A. \cos^2 \tfrac{1}{2} B + \cos^2 \tfrac{1}{2} A. \cos^2 \tfrac{1}{2} C + + \cos^2 \tfrac{1}{2} B. \cos^2 \tfrac{1}{2} C + \tfrac{1}{4} [\sin A. \sin B + \sin A. \sin C + \sin B. \sin C.]},$$

unde substitutis quantitatibus illis notis, per trianguli latera expressis, quas $\cos \tfrac{1}{2} A$, $\cos \tfrac{1}{2} B$, $\cos \tfrac{1}{2} C$, $\sin A$ etc. aequant, si brevitatis gratia, $s = \tfrac{1}{2}(a+b+c)$ esse, ponamus:

$$\Delta\, DEF = \frac{\frac{a+b+c}{abc}\,.\,r^2\,\Delta^2}{1-\frac{s}{abc}[a(s-a)+b(s-b)+c(s-c)]+\frac{s}{abc}\Big[\frac{s(s-a)(s-b)}{c}+\frac{s(s-a)(s-c)}{b}+\frac{s(s-b)(s-c)}{a}\Big]+\frac{s.(s-a(s-b(s-c)}{abc}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)};$$

ergo, quum nunquam esse nequeat

$$a(s-a)+b(s-b)+c(s-c)=2[(s-a)(s-b)+(s-a)(s-c)+(s-b)(s-c)]=$$
$$(s-a)(s-b)+(s-a)(s-c)+(s-b)(s-c)+(s-a).(s-b)+(s-a)(s-c)+(s-b)(s-c),$$

$$\Delta\, DEF = \frac{2r^2\Delta}{\frac{abc}{s}+(s-a)(s-b)(s-c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-(s-a)(s-b-(s-a)(s-c)-(s-b)(s-c)+(s-a)(s-b)(s-c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$$

$$= \frac{2r^2\Delta}{\frac{abc}{s}+2(s-a)(s-b)(s-c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-(s-a)(s-b)-(s-a)(s-c)-(s-b)(s-c)},$$

unde, quoniam est

$$s[(s-a)(s-b)+(s-a)(s-c)+(s-b)(s-c)]=(s-a)(s-b)(s-c)+abc,$$

$$\Delta\, DEF = \frac{2s.\,r^2\Delta}{abc+2s(s-a)(s-b)(s-c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-(s-a)(s-b)(s-c)-abc}$$

$$= \frac{(a+b+c)\,r^2\Delta}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)2\Delta^2-\frac{2\Delta^2}{a+b+c}}$$

$$= \frac{(a+b+c)^2\,r^2.\,\Delta}{\left[(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-1\right]2\Delta^2}$$

$$= \frac{2\Delta}{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-1};$$

quae quidem propositio satis memorabilis minus commode verbis enunciari potest.

§. 19.

Tertiae, ad quam nunc progredimur, transversarum ternioni quamquam haud desunt proprietates memoratu satis dignae, tamen, quum jam ab aliis, inprimis vero a cl. Feu-

erbachio, optime explicitae sint, immorari iis non adtinet. Una est propositio, quam me unquam legere cum non meminerim tradere nunc liceat. Quoniam nimirum est

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \quad ^{12)}$$

habemus

$$\frac{ab+ac+bc}{\sin A.\sin B.\sin C} = \frac{4R^2}{\sin A} + \frac{4R^2}{\sin B} + \frac{4R^2}{\sin C}$$
$$= 4R^2\left[\frac{\sin A.\sin B + \sin A.\sin C + \sin B.\sin C}{\sin A.\sin B.\sin C}\right];$$

ergo

$$ab+ac+bc = 4R^2\ [\sin A.\sin B + \sin A.\sin C + \sin B.\sin C]$$
$$= 4R^2\left[\frac{4\Delta^2}{abc^2} + \frac{4\Delta^2}{ab^2c} + \frac{4\Delta^2}{a^2bc}\right]$$
$$= 4R^2.\frac{2\Delta}{abc}\left[\frac{2\Delta}{c} + \frac{2\Delta}{b} + \frac{2\Delta}{a}\right]$$
$$= 2R\ [\overset{(*)}{d} + \overset{(*)}{e} + \overset{(*)}{f}]$$

h. e. rectangulorum, a binis trianguli lateribus factorum, summa rectangulum id aequat, quod a diametro circuli exterioris (triangulo circumscripti) et perpendiculorum, a verticibus in latera demissorum, summa ortum est.

§. 20.

Quartam, quae jam sequitur, transversarum ternionem voluimus esse eam, qua angulorum fragmenta A', B', C' inter se efficiuntur aequalia.

Primum autem, quum neque ex elementis cognitum sit, neque per se satis appareat, cujusnam constructionis ope ejusmodi transversae sint ducendae, eam explicabimus, quae facillima ceterisque omnibus praeferenda esse nobis videtur.

Describantur tres circuli (Fig. 9.), quorum quisque duorum trianguli laterum altero tanquam chorda, altero eoque posteriori tanquam tangente utitur; qui quidem uno eodemque puncto, quo et nostrae ternionis transversae, invicem secentur necesse est.

Binos enim ejusmodi circulos sese transire per se intelligitur. Itaque si punctum G illud est, quod praeter alterum C circuli ACP et BCO una transeunt, per idem punctum et tertium circulum pervadere contendimus. Nam qui circulus ita describitur, ut puncta A, B, G sint in ejus circuitu, trianguli latus c chordam habere patet; idem

[12]) Simplicissima hujus theorematis demonstratio est haec:

$$\frac{CD}{CG} = \frac{a}{2R} = \sin CGD = \sin \tfrac{1}{2} BGC = \sin A, \text{ ergo } 2R = \frac{a}{\sin A} \text{ etc. (Fig. 8.)}$$

vero, quum ex constructione priorum circulorum omnino sit

$$< BAG = < ACG = < CBG,$$

a latere a tangitur, eamque ob caussam ab eo, quem nos innuimus, circulo non discrepat; unde vera esse, quae diximus, patet.

§ 21.

Angulos, ad quos transversae in puncto G secantur, angulis A, B, C esse aequales, scilicet $\alpha = A$, $\beta = B$, $\gamma = C$, facile intelligitur.

Quod triangulum HIK demissis a puncto G in latera trianguli primarii perpendiculis eorumque finibus inter se junctis nascitur, non solum triangulo ABC simile est, sed etiam ita comparatum, ut quartae ternionis transversae et ipsae puncto G invicem secentur.

Quum enim quadrilatera $AIGK$, $BHGK$ et $CHGI$ ad ea sint referenda, quibus circulus circumscribi potest, esse:

$$< KIG = < KAG = < ACF,$$

nec minus $< HIG = < GCH$, et hanc ob caussam

$$< HIK = C$$

facili negotio perspicitur; similique ratione esse videmus:

$$< KHI = B \;;\; < HKI = A\;;$$

$$< KIG = < BAH = < CBE = < HKG = < ACF = < GHI,$$

ex quo illum ipsum, quod volumus, esse patet.

Idem si nunc in triangulo HIK fieret, quod modo in primario ABC factum est, tertium sane prodiret triangulum, utrique praecedenti simile, cui et ipsi punctum G id esset, in quo quartae ternionis transversae invicem sese transirent. Itaque hanc constructionem iterum iterumque in usum vocando infinita triangulorum, inter se similium, quibus punctum G commune quasi centrum esset, multitudo fieri posset.

§. 22.

In omni triangulo, dejectis a quolibet puncto $v.\ c.\ G$ in latera perpendiculis, GH, GI, GK, esse omnino:

$$a.\,BH + b.\,CI + c.\,AK = a.\,CH + b.\,AI + c.\,BK = \tfrac{1}{2}(a^2 + b^2 + c^2),$$

quum in elementis geometriae demonstretur, habemus:

$$\tfrac{1}{4}(a^2+b^2+c^2)\,\mathrm{tg}\,\overset{(*)}{A'} = \tfrac{1}{2}\,[a.\,BH + b.\,CI + c.\,AK]\;\mathrm{tg}\,\overset{(*)}{A'}$$
$$= \tfrac{1}{2}\,[a.\,GH + b.\,GI + c.\,GK]$$
$$= \triangle,$$

unde, quoniam

$$\text{cotg } A + \text{cotg. } B + \text{cotg } C = \frac{a^2 + b^2 + c^2}{4\,\Delta}$$

esse goniometria docet,

$$\text{cotg } \overset{(4)}{A'} = \frac{a^2 + b^2 + c^2}{4\,\Delta} = \text{cotg } A + \text{cotg } B + \text{cotg } C\ ^{13)};$$

h. e. angulorum aequalium, ad quos ternionis quartae transversae trianguli latera secant, quantitas ita comparata est, ut cujusque cotangens angulorum A, B, C cotangentes, ad summam redactas, aequet.

Cor. 1. Facile hinc colligimus esse:

1) $\text{cosec}^2 \overset{(4)}{A'} = \text{cosec}^2 A + \text{cosec}^2 B + \text{cosec}^2 C$

2) Si angulorum A, B, C unus v. c. A recto est aequalis,

$$\text{cotg } \overset{(4)}{A'} = \text{cotg } B + \text{cotg } C = 2 \text{ cosec } 2\,B = 2 \text{ cosec } 2\,C = \text{cosec } 2\,B + \text{cosec } 2\,C$$

§. 23.

Triangula BGC et HGI inter se sunt similia; inde

$$a : GC :: HI : GI;$$

ergo $$HI = a.\,\frac{GI}{GC} = a.\,\sin \overset{(4)}{A'}$$

et hanc ob caussam,

$$\Delta : \Delta\, HIK :: a^2 : a^2 .\, \sin^2 \overset{(4)}{A'} :: 1 : \sin^2 \overset{(4)}{A'}$$
$$:: \text{cosec}^2 A' : 1$$
$$:: \text{cosec}^2 A + \text{cosec}^2 B + \text{cosec}^2 C : 1$$

Coroll. Itaque si utrumque triangulum est rectangulum et aequicrurum, hanc nanciscimur proportionem:

$$\Delta : \Delta\, HIK :: 1 + 2 \text{ cosec}^2 \tfrac{1}{2} R : 1 :: 5 : 1$$

h. e. triangulum primarium ejus, quod ei inscriptum est, trianguli HIK aream quinquies complectitur.

§. 24.

Propter triangulorum BGD et BAD, CGE et CBE, AGF et ACF, similitudinem sit

$$\overset{(4)}{a'^2} = d.\,\overset{(4)}{d''}$$

similique ratione

$$\overset{(4)}{b'^2} = \overset{(4)}{e.e''},\ \overset{(4)}{c'^2} = \overset{(4)}{f.f''}$$

[13]) Elegantissimum hoc theorema debemus cl. Crellio, qui plures, a nostra satis discrepantes, ejus demonstrationes tradidit in libro laudato.

necesse est; inde

$$\overset{(4)}{d}.e.f : \overset{(4)}{a'}.b'.c' :: \overset{(4)}{a'}.b'.c' : \overset{(4)}{d''}.e''.f''$$

h. e. parallelepipedum rectum, a tribus laterum fragmentis non contiguis factum, solidum est geometricae proportionis medium inter parallelepipida ea, quorum alterum a transversis ipsis, alterum ab earum fragmentis inferioribus oritur.

§. 25.

Eadem ratione habemus:

$$\overset{(4)}{a'} : e' :: \overset{(4)}{d} : c$$
$$b' : f' :: e : a$$
$$c' : d' :: f : b,$$

ergo $$\overset{(4)}{a'}.b'.c' : \overset{(4)}{d'}.e'.f' :: \overset{(4)}{d}.e.f : a.b.c$$

adeoque $$\overset{(4)}{d'}.e'.f' : \overset{(4)}{d''}.e''.f'' :: a.b.c : a'.b'.c' \quad (§. 24.).$$

Est autem, uti cl. Crellius demonstravit:

$$\overset{(4)}{a'} = a.\frac{c^2}{a^2+c^2},\ \overset{(4)}{b'} = b.\frac{a^2}{a^2+b^2},\ \overset{(4)}{c'} = c.\frac{b^2}{b^2+c^2};$$

proinde

$$\overset{(4)}{d'}.e'.f' : \overset{(4)}{d''}.e''.f'' :: a.b.c : \frac{a^3.b^3.c^3}{(a^2+b^2)(a^2+c^2)(b^2+c^2)}$$

$$d'.e'.f' : d''.e''.f'' :: (a^2+b^2)(a^2+c^2)(b^2+c^2) : a^2.b^2.c^2;$$

quam quidem proportionem eo potissimum consilio attulimus, ut cum ea, quae supra (§. 13.) legitur, comparari posset; qua comparatione hac quoque proportiones dignae sunt:

$$\overset{(1)}{d'}.e'.f' : \overset{(1)}{d''}.e''.f'' :: a.b.c : \overset{(1)}{a'}.b'.c'$$
$$\overset{(2)}{d'}.e'.f' : \overset{(2)}{d''}.e''.f'' :: a.b.c : \overset{(2)}{a'}.b'.c' \quad (§. 14.)$$
$$\overset{(3)}{d'}.e'.f' : \overset{(3)}{d''}.e''.f'' :: a.b.c : \overset{(3)}{a'}.b'.c' \quad ^{14})$$
$$\overset{(4)}{d'}.e'.f' : \overset{(4)}{d''}.e''.f'' :: a.b.c : \overset{(4)}{a'}.b'.c'$$
$$\overset{(5)}{d'}.e'.f' : \overset{(5)}{d''}.e''.f'' :: a.b.c : \overset{(5)}{a'}.b'.c' \quad (§. 28.)$$

[14]) hanc proportionem esse veram, e triangulorum AGF, ABD etc. similitudine facile colligitur.

§. 26.

Quodsi triangulo ABC circulus circumscribitur, cujus extremitati transversae, quae producuntur, in punctis P, Q, R iterum occurrunt, ductis lineis AR, BP etc., non solum

$$AR = BP = CQ,$$

sed etiam triangula AGQ, AGR, BGP, BGR, CGP, CGQ et inter sese ipsa, et triangulo primario similia esse facile adparet; est igitur:

$$AG : BG :: AQ : BP$$
$$BG : CG :: BR : CQ$$
$$CG : AG :: CP : AR$$

ergo $AQ.\, BR.\, CP = AR.\, BP.\, CQ = \overline{AR}^3$,

ex quo concludere licet, quae transversae in triangulo, cujus latera sunt $AR + AQ$, $AR + BR$, $AR + CP$, ita ducuntur, ut sit $a' = b' = c' = AR$, vel $a'' = b'' = c'' = AR$, has in uno puncto sese invicem omnino transire.

§. 27.

Quinta transversarum ternio, quae angulos A, B, C ita secat, ut sit $A'' = B'' = C''$, proxime ad praecedentem accedit. Primum autem hoc jam in eo cernitur, quod constructio, cujus ope lineae nostrae duci possunt, ei, quam supra (§. 20. Fig. 12) tradidimus, est plane simillima. Qui enim circuli tres ita describuntur, ut laterum duorum trianguli altero tanquam chorda, altero, eoque priori, tanquam tangente quisque utatur, hos uno eodemque, quo et ternionis nostrae transversae, puncto sese invicem transire eo, quo antea factum est, modo probatur. Deinde nunc quoque non solum angulos, ad quos transversae secantur, angulis A, B, C, singulum singulo, aequales, sed etiam omnino esse:

$$\overset{[5]}{\cot A''} = \cot A + \cot B + \cot C$$

facile perspicitur; denique eadem, quam cl. Crellius secutus est, via, esse

$$\overset{[5]}{a'} = a\,\frac{a^2}{a^2 + b^2},\quad \overset{[5]}{b'} = b\,\frac{b^2}{b^2 + c^2},\quad \overset{[5]}{c'} = c\,\frac{c^2}{a^2 + c^2}$$

$$\overset{[6]}{a''} = a.\,\frac{b^2}{a^2 + b^2},\quad \overset{[5]}{b''} = b.\,\frac{c^2}{b^2 + c^2},\quad \overset{[6]}{c''} = c.\,\frac{a^2}{a^2 + c^2},$$

demonstrari potest.

§. 28.

His autem constitutis, aequationes sequentes veras esse facile intelligitur:

1) $\overset{(4)}{A'} = \overset{(4)}{B'} = \overset{(4)}{C'} = \overset{(4)}{A''} = \overset{(4)}{B''} = \overset{(4)}{C''}$

2) $a'.\overset{(4)}{b'}.c' = a''.\overset{(4)}{b''}.c'' = \frac{a^3.\ b^3.\ c^3}{(a^2+b^2)\ (a^2+c^2)\ (b^2+c^2)} = a'.\overset{(5)}{b'}.c' = a''.\overset{(5)}{b''}.c''$

h. e. parallelepipeda recta, a ternis laterum fragmentis non contiguis in utraque ternione facta, invicem aequantur.

3) $d.\overset{(4)}{e}.f = d.\overset{(5)}{e}.f$, quoniam

$d.\overset{(4)}{e}.f : a'.\overset{(4)}{b'}.c' :: \sin A. \sin B. \sin C : \sin^3 A'$

$d.\overset{(5)}{e}.f : a''.\overset{(5)}{b''}.c'' :: \sin A. \sin B. \sin C : \sin^3 A''$,

esse patet;

h. e. parallelepipeda recta a transversis utriusque ternionis facta inter se sunt aequalia.

4) $d''.\overset{(4)}{e''}.f'' = d''.\overset{(5)}{e''}.f''$, quum esse oporteat:

$d.\overset{(4)}{e}.f : a'.\overset{(4)}{b'}.c' :: a'.\overset{(4)}{b'}.c' : d''.\overset{(4)}{e''}.f''$

et $d.\overset{(5)}{e}.f : a''.\overset{(5)}{b''}.c'' :: a''.\overset{(5)}{b''}.c'' : d''.\overset{(5)}{e''}.f''$

5) $d'.\overset{(4)}{e'}.f' = d'.\overset{(5)}{e'}.f'$.

Si enim *AD*, *BE*, *CF* (Fig. 13.) sunt quartae ternionis transversae, lineae vero *AH*, *BI*, *CK* ad quintam referendae, has proportiones locum habere necesse est:

$\overset{(4)}{d'} : b :: \sin \overset{(4)}{A'} : \sin A$

$\overset{(5)}{d'} : c :: \sin \overset{(5)}{A''} : \sin A$,

ergo $\overset{(4)}{d'} : \overset{(\ )}{d'} :: b : c$

similique ratione $\overset{(4)}{e'} : \overset{(5)}{e'} :: \overset{(4)}{c} : \overset{(5)}{a}$

$\overset{(4)}{f'} : \overset{(5)}{f'} :: a : b$,

unde, $\overset{(4)}{d'.e'.f'} = \overset{(5)}{d'.e'.f'}$;

ergo parallelepipeda recta et a superioribus et ab inferioribus transversarum fragmentis in utraque ternione nostra facta ejusdem sunt magnitudinis.

Cor. E proportione modo tradita:

$\overset{(4)}{d'} : \overset{(5)}{d'} :: b : c$,

in omni triangulo aequicruro fragmenta superiora duarum transversarum, quae ad quartam quintamque ternionem referuntur, et a vertice basi opposito ducuntur, inter se aequalia esse, jure colligimus.

§. 29.

Quodsi duo triangula similia ABC, et $\mathfrak{ABC}$ ita sunt comparata, ut sit

$$d'.\,e'.\,f' = \mathfrak{d}'.\,\mathfrak{e}'.\,\mathfrak{f}'$$

eadem sibi invicem congruant necesse est. Vidimus enim, esse omnino

$$\overset{(4)}{d'.\,e'.\,f'} = a.\,b.\,c\,.\;\frac{\sin^3 \overset{(4)}{A'}}{\sin A.\,\sin B.\,\sin C}$$

$$\overset{(5)}{\mathfrak{d}'.\,\mathfrak{e}'.\,\mathfrak{f}'} = \mathfrak{a}.\,\mathfrak{b}.\,\mathfrak{c}\,.\;\frac{\sin^3 \mathfrak{A}''}{\sin \mathfrak{A}.\,\sin \mathfrak{B}.\,\sin \mathfrak{C}};$$

ergo, quum sit $\overset{(4)}{d'.\,e'.\,f'} = \overset{(5)}{\mathfrak{d}'.\,\mathfrak{e}'.\,\mathfrak{f}'}$ ex hyp.

et $\overset{(4)}{A'} = \overset{(5)}{\mathfrak{A}''}$, (§. 28.)

fieri non potest, quin habeamus $a.\,b\;c = \mathfrak{a}.\,\mathfrak{b}.\,\mathfrak{c}$

adeoque $a = \mathfrak{a}\,,\; b = \mathfrak{b}\,,\; c = \mathfrak{c}.$

Nam si unquam

$$a > \mathfrak{a}$$

esse posset, propter triangulorum similitudinem, et $b > \mathfrak{b}$, $c > \mathfrak{c}$ esse oporteret, quod quidem aequationi

$$a.\,b.\,c = \mathfrak{a}.\,\mathfrak{b}.\,\mathfrak{c}$$

plane repugnat; eadem vero de caussa quum fieri nequeat, ut sit $a < \mathfrak{a}$, triangula ABC, $\mathfrak{ABC}$ sibi congruere recte concluditur.

§. 30.

Utriusque ternionis (quartae et quintae) transversis una ductis, tria existunt triangula aequicrura, AMB, BLC, ANC, quorum areae, in unam summam collectae, primarii aream aequant. Patet enim esse:

$$\triangle\, AMB = \tfrac{1}{2}\, c.\, MR = \tfrac{1}{4}\, c^2.\ \text{tg}\, \overset{(4)}{A'}$$

$$\triangle\, BLC = \tfrac{1}{4}\, a^2.\ \text{tg}\, \overset{(4)}{A'}$$

$$\triangle\, ANC = \tfrac{1}{4}\, b^2.\ \text{tg}\, \overset{(4)}{A'};$$

ergo

$$\triangle\, AMB + \triangle\, BLC + \triangle\, ANC = \tfrac{1}{4}\,(a^2 + b^2 + c^2)\ \text{tg}\, \overset{(4)}{A'} = \triangle\ (\S.\,22.)$$

Cor. Ex quo esse colligimus:

$$\triangle\, CGL = \triangle\, AGN + \triangle\, BGM$$
$$\triangle\, CON = \triangle\, AOM + \triangle\, BOL\,.$$

§. 31.

Sextam, de qua dicturi sumus, transversarum ternionem adpellavimus eam, quae trianguli latera ita secat, ut sit

$$a' = c'', \; b' = a'', \; c' = b''.$$

Primum autem quum ex elementis satis constet, latera trianguli ab his transversis iisdem punctis secari, quibus circulus, triangulo inscriptus, ipsa tangit, quisque, quid ternionis nostrae transversas ducturo faciendum sit, facile videt. Deinde non minus adparet, esse omnino:

$$\overset{(vi)}{a'} = \overset{(vi)}{c''} = \tfrac{1}{2}(a-b+c); \; \overset{(vi)}{b'} = \overset{(vi)}{a''} = \tfrac{1}{2}(a+b-c); \; \overset{(vi)}{c'} = \overset{(vi)}{b''} = \tfrac{1}{2}(-a+b+c);$$

ex quo cogitur, esse:

$$a'.\overset{(vi)}{b'}.c' = a''.\overset{(vi)}{b''}.c'' = \tfrac{1}{8}(-a+b+c)(a-b+c)(a+b-c) = \frac{2\,\Delta^2}{a+b+c}$$
$$= \text{r.}\,\Delta$$

h. e. parallelepipedum rectum, a tribus laterum fragmentis non contiguis factum, aequale est prismati ei, cujus basis triangulum ipsum, altitudo autem circuli interioris radium aequat.

§. 32.

Transversarum finibus *D*, *E*, *F*, (Fig. 14.) inter se junctis, quod fit triangulum, ita esse comparatum, ut sit

$$\text{r.}\,\Delta = 2\,R.\;\Delta\;DEF,$$

quum cl. Feuerbachius demonstraverit [15]), hinc esse

$$a'\,\overset{(vi)}{b'}\,c' = a''.\overset{(vi)}{b''}.c'' = 2\,R.\,\Delta\,D\overset{(vi)}{E}F$$

sequitur; h. e. eadem illa, quae modo memoravimus, parallelepipeda recta, prismati aequantur ei, cujus basis est triangulum primario inscriptum, altitudo autem exterioris circuli diametrus.

§. 33.

Quoniam omnino est:

$$(-a+b+c)(a-b+c) + (-a+b+c)(a+b-c) + (a-b+c)(a+b-c) = a(-a+b+c)$$
$$+ b(a-b+c) + c(a+b-c),$$

hanc nanciscimur aequationem:

$$a'.b' + a'.\overset{(vi)}{c'} + b'.c' = a''.b'' + a''.\overset{(vi)}{c''} + b''.c'' =$$
$$\tfrac{1}{4}(-a+b+c)(a-b+c) + \tfrac{1}{4}(-a+b+c)(a+b-c) + \tfrac{1}{4}(a-b+c)(a+b-c) =$$

[15]) in l. l. §. 8.

$$\tfrac{1}{4}\,[a\,(-a+b+c)+b\,(a-b+c)+c\,(a+b-c)]=\tfrac{1}{2}\,[a\,c'+b\,\overset{(\circ)}{a'}+c\,b']$$

h. e. rectangulorum a binis laterum partibus non contiguis effectorum summa dimidiam eorum summam aequat, quorum singulum quodvis a trianguli laterum uno et prioris parte huic non adjacente orta sunt.

§. 34.

Non minus hanc aequationem

$$(-a+b+c)\,(a-b+c)+(-a+b+c)\,(a+b-c)+(a-b+c)\,(a+b-c)=2\,(ab+ac+bc)$$
$$-(a^2+b^2+c^2)$$

veram esse apparet; hinc autem colligimus:

$$a'\,b'+a'\,\overset{(\circ)}{c'}+b'.\,c'=a''.\,b''+a''.\,\overset{(\circ)}{c''}+b''.\,c''=r\,(r'+r''+r''')$$ [10].

h. e. rectangulorum a binis laterum partibus non contiguis factorum summa rectangulum id aequat, quod ab interioris circuli radio et reliquorum, trianguli latera contingentium, circulorum radiis in unam summam collectis ortum est.

§. 35.

Fieri quum nequeat, quin sit

$$a\,(a-b+c)+b\,(a+b-c)+c\,(-a+b+c)=a\,(a+b-c)+b\,(-a+b+c)+c\,(a-b+c)$$
$$=a^2+b^2+c^2,$$

sit etiam necesse est

$$a.\,a'+b.\,\overset{(\circ)}{b'}+c.\,c'=a.\,a''+b.\,b''\overset{(\circ)}{+}c.\,c''=\tfrac{1}{2}\,(a^2+b^2+c^2)$$

h. e. rectangula, quorum cuique singulo singulum trianguli latus alterumque ipsius fragmentum sunt latera, si in unam summam colliguntur, rectangulorum ab iisdem lateribus singulis et altero cujusque fragmento factorum summam aequant.

Cor. Quum per se satis appareat, esse

$$a.\,a'+b.\,\overset{(1)}{b'}+c.\,c'=a.\,a''+b.\,\overset{(1)}{b''}+c.\,c''=\tfrac{1}{2}\,(a^2+b^2+c^2)$$

nec minus ex propositione, in elementis obvia, cujus jam supra (§. 22.) mentionem fecimus,

$$a.\,a'+\overset{(2)}{b}.\,b'+c.\,c'=a.\,a''+b.\,\overset{(2)}{b''}+c.\,c''=\tfrac{1}{2}\,(a^2+b^2+c^2),$$

omnino esse videmus

$$a.\,a'+\overset{(1)}{b}.\,b'+c.\,c'=a.\,a''+b.\,\overset{(1)}{b''}+c.\,c''=a.\,a'+b.\,\overset{(2)}{b'}+c\,c'=$$
$$a.\,a''+\overset{(3)}{b}.\,b''+c\,c''=a.\,a'+b.\,\overset{(5)}{b'}+c\,c'=a.\,a''+b.\,\overset{(6)}{b''}+c\,c''=$$
$$a\,a'+b.\,\overset{(7)}{b'}+c\,c'=a\,a''+b.\,\overset{(7)}{b''}+c.\,c''\quad(\text{§. }38.)$$

[10]) cf. Feuerbachius in l. l. §. 6.

§. 36.

Unusquisque, harum rerum cognitionis non plane expers, facile intelligit, esse
$abc - a(-a+b+c)^2 - b(a-b+c)^2 - c(a+b-c)^2 = (-a+b+c)(a-b+c)(a+b-c)$;
quam ob rem habemus

$$a\,c'^2 + b.\overset{(\circ)}{a'^2} + c.\,b'^2 = a\,b\,c - \tfrac{1}{4}(-a+b+c)(a-b+c)(a+b-c)$$
$$= 4R\,\Delta - \frac{4\,\Delta^2}{a+b+c} = (2R-r)\,2\,\Delta \quad .$$

h. e. parallelepipeda recta, quorum cuique singulo singulum trianguli latus est altitudo, basis vero quadratum super lateris prioris fragmento non vicino constructum, si ad summam rediguntur, prismati aequalia sunt illi, cujus basis trianguli ipsius aream bis complectitur, altitudo autem lineam eam aequat, qua diametrus exterioris circuli radium interioris excedit.

Cor. His demonstratis, facile perspicitur, esse

$$a.c'^2 + b.\overset{(\circ)}{a'^2} + c.b'^2 + 2\,a'.b'.c' + 2 = a.b''^2 + b.\overset{(\circ)}{c''^2} + c.a''^2 + a''.b''.c'' = 4R.\Delta = a.b.c$$

§. 37.

Quemadmodum circulus *DEF* (Fig. 14.) triangulo primario inscriptus est, ita triangulum *DEF* circulo *DEF*; sit igitur

$$DE.\,DF.\,EF = 4\,r\,\Delta$$

necesse est, eamque ob caussam

$$DE.\,\overset{(\circ)}{DF}.\,EF = 4\,a'.\overset{(\circ)}{b'}.\,c' \quad (§.\ 31.)$$

h. e. parallelepipedum, a lineis transversarum fines jungentibus ortum, parallelepipedi a tribus laterum fragmentis non contiguis facti quadruplum aequat.

§. 38.

Septima, quae jam nobis relinquitur, transversarum ternio, ita comparata, ut sit

$$\overset{(')}{a'} = \overset{(')}{b''},\ \overset{(')}{b'} = \overset{(')}{c''},\ \overset{(')}{c'} = \overset{(')}{a''},$$

arctissime cum praecedenti cohaeret. Facile enim perspicitur, esse

$$\overset{(')}{a'} = \tfrac{1}{2}(a+b-c),\ \overset{(')}{b'} = \tfrac{1}{2}(-a+b+c),\ \overset{(')}{c'} = \tfrac{1}{2}(a-b+c)$$

et hanc ob caussam

$$\overset{(')}{a'} = \overset{(\circ)}{a''},\ \overset{(')}{b'} = \overset{(\circ)}{b''},\ \overset{(')}{c'} = \overset{(\circ)}{c''}$$

h: e. altera harum ternionum in alteram mutatur cujusque lateris fragmentis inter se permutatis. Itaque huic ternioni tractandae immorari non opus est.

§. 39.

Jam vero addere sit nobis licitum aliam propositionem, quae lineas transversas, in eodem puncto sese transeuntes spectat.

Quodsi a trianguli cujusdam verticibus A, B, C (Fig. 15.), lineae AI, AK, HB, BK, CH, CI ita ducuntur, ut sit

$$< BAK = < CAI; \quad < ABK = < CBH; \quad < BCH = < ACI,$$

lineae AH, BI, CK eodem puncto invicem secentur necesse est.

Patet enim esse:

$$\begin{aligned} AI &: IE :: \sin AEI : \sin \varphi \\ IE &: CI :: \sin \varphi'' : \sin IEC \\ CH &: HD :: \sin CDH : \sin \varphi'' \\ HD &: BH :: \sin \varphi' : \sin BDH \\ BK &: FK :: \sin BFK : \sin \varphi' \\ FK &: AK :: \sin \varphi : \sin AFK, \end{aligned}$$

ergo $$AI.\ BK.\ CH = AK.\ BH.\ CI.$$

Simili autem ratione habemus:

$$\begin{aligned} BI &: CI :: \sin (C + \varphi'') : \sin B' \\ AI &: BI :: \sin B'' : \sin (A + \varphi) \\ CH &: AH :: \sin A'' : \sin (C + \varphi'') \\ AH &: BH :: \sin (B + \varphi') : \sin A' \\ CK &: AK :: \sin (A + \varphi) : \sin C' \\ BK &: CK :: \sin C'' : \sin (B + \varphi'), \end{aligned}$$

ergo

$$AI.\ BK.\ CH : AK.\ BH.\ CI :: \sin A''.\ \sin B''.\ \sin C'' : \sin A'.\ \sin B'.\ \sin C'$$

adeoque, quum sit

$$AI.\ BK.\ CH = AK.\ BH.\ CI,$$
$$\sin A'.\ \sin B'.\ \sin C' = \sin A''.\ \sin B''.\ \sin G'';$$

transversae AH, BI, CK igitur idem punctum transeant necesse est (§. 3. Cor.).

Schol. Enunciationem nostram non minus tum quoque se recte habere, cum lineae AI, AK etc. ab interiori trianguli laterum parte ducantur, ipsa demonstrationis natura luculenter docet.

Coroll. 1. Quum anguli, φ, φ', φ'' arbitrariae sint magnitudinis, omnino fieri posse patet, ut sit

$$\varphi + \tfrac{1}{2} A = \varphi' + \tfrac{1}{2} B = \varphi'' + \tfrac{1}{2} C = 90^\circ;$$

deinde vero binas lineas AI, AK; BH, BK et CH, CI unam rectam efficere, vel triangulum HIK existere, per se apparet.

Ex iis autem, quae sumsimus,

$$\varphi + \tfrac{1}{2} A + \varphi' + \tfrac{1}{2} B + \varphi'' + \tfrac{1}{2} C = 3 R,$$

et hanc ob caussam

$$\varphi + \varphi' + \varphi'' = 2 R$$

esse sequitur. Praeterea est

$$\varphi + \varphi' + K = \varphi + \varphi'' + I = \varphi' + \varphi'' + H = 2 R,$$

unde

$$\varphi = H, \varphi' = I, \varphi'' = K,$$

ideoque quadrangula $ABHI$, $ACHK$, et $BCIK$ ita esse comparata, ut circuli iis circumscribi possint, colligimus. Ex quo cogitur, esse

$$< CAH = < HKC = < BIC = < BAH = \tfrac{1}{2} A;$$

ergo $< HAK = < HAI = 90°$ similique ratione

$$< IBH = < IBK = < KCH = < KCI;$$

h. e. transversae ad angulos rectos trianguli HIK lateribus occurrunt.

Coroll. 2. Quum nihil plane intersit, utrum triangulorum ABC, HIK tanquam primarium et primigenium spectetur, sponte nobis hinc prodit hoc theorema: in omni triangulo acutiangulo punctum illud, quo perpendicula a verticibus ad latera demissa secantur, centrum est ejus circuli, qui triangulo, quod perpendiculorum finibus inter se junctis extitit, inscribi potest.

Plura alia ejus modi hic proferri possent, sed non adtinet in his rebus tractandis, quae satis notae sunt, perdere tempus.

Cor. Angulos, ad quos transversae sese transeunt, angulis φ, φ', φ'', singulum singulo, nunc aequales esse patet.

Sectio commentationis secunda.

De lineis transversis, generatim spectatis.

§. 40.

Redeamus jam eo, unde egressi sumus, ad illud scilicet transversarum genus, quod ita comparatum est, ut singulae cujusque ternionis lineae non idem punctum una transeant, sed triangulum quoddam efficiant.

Primum autem, ejusmodi triangulum HIK (Fig. 17 et 18.) primario esse simile, ubi sit

$$\text{aut } A' = B' = C'$$
$$\text{aut } A'' = B'' = C''$$

facili negotio intelligitur; nec minus id adparet, aream hujus trianguli, primario similis, pro diversa anguli A' vel A'' magnitudine fieri diversam, vel, quod geometrae dicunt, illam trianguli aream anguli A' vel A'' esse functionem. Cujus quidem functionis formam paullo accuratius explorare jam studeamus.

Est autem

$$\triangle ABH = \tfrac{1}{2} c.\ AH.\ \sin A'$$

et $$AH : c :: \sin(B - A') : \sin B,$$

unde $$\triangle AHB = \tfrac{1}{2} c^2.\ \frac{\sin(B - A').\sin A'}{\sin B}.$$

Simili ei, quam modo inivimus, via praeterea has aequationes nanciscimur:

$$\triangle AIC = \tfrac{1}{2} b^2.\ \frac{\sin(A - A').\sin A'}{\sin A}$$

$$\triangle BCK = \tfrac{1}{2} a^2.\ \frac{\sin(C - A').\sin A'}{\sin C},$$

ergo, quum

$$\triangle HIK = \triangle - \triangle ABH - \triangle ACI - \triangle BCK$$

esse pateat,

$$\triangle GHI = \triangle - \tfrac{1}{2} a^2.\ \frac{\sin A'.\sin(C - A')}{\sin C} - \tfrac{1}{2} b^2.\ \frac{\sin A'.\sin(A - A')}{\sin A} - \tfrac{1}{2} c^2.\ \frac{\sin A'.\sin(B - A')}{\sin B}$$

$$= \triangle - \tfrac{1}{2} \sin A' \Big[a^2 (\cos A' - \sin A'.\ \text{cotg.}\ C) + b^2 (\cos A' - \sin A'.\ \text{cotg}\ A) - - c^2 (\cos A' - \sin A'.\ \text{cotg}\ B) \Big]$$

$$= \triangle - \tfrac{1}{2} \sin^2 A' \left[a^2 (\text{cotg}\ A' - \text{cotg}\ C) + b^2 (\text{cotg}\ A' - \text{cotg}\ A) + c^2 (\text{cotg}\ A' - \text{cotg}\ B) \right]$$

$$= \triangle - \tfrac{1}{2} \sin^2 A' \Big[(a^2 + b^2 + c^2)(\text{cotg}\ A' - \text{cotg}\ A - \text{cotg}\ B - \text{cotg}\ C) + a^2 (\text{cotg}\ A + \text{cotg}\ B) + b^2.(\text{cotg}\ B + \text{cotg}\ C) + c^2 (\text{cotg}\ A + \text{cotg}\ C) \Big].$$

§. 41.

In omni triangulo sit

$$a^2 (\text{cotg}\ A + \text{cotg}\ B) + b^2 (\text{cotg}\ B + \text{cotg}\ C) + c^2 (\text{cotg}\ A + \text{cotg}\ C) = \frac{ab}{\sin C} + \frac{ac}{\sin B} + \frac{bc}{\sin A}$$

necesse est.

Habemus enim

$$a^2 (\text{cotg}\ A + \text{cotg}\ B) + b^2 (\text{cotg}\ B + \text{cotg}\ C) + c^2 (\text{cotg}\ A + \text{cotg}\ C) =$$

$$a^2.\left[\frac{\cos A.}{\sin A} + \frac{\cos B}{\sin B}\right] + b^2 \left[\frac{\cos B}{\sin B} + \frac{\cos C}{\sin C}\right] + c^2 \left[\frac{\cos A}{\sin A} + \frac{\cos C}{\sin C}\right] =$$

$$a^2.\ \frac{\sin(A+B)}{\sin A.\sin B} + b^2.\ \frac{\sin(B+C)}{\sin B.\sin C} + c^2.\ \frac{\sin(A+C)}{\sin A.\sin C} =$$
$$a^2.\ \frac{\sin C}{\sin A.\sin B} + b^2.\ \frac{\sin A}{\sin B.\sin C} + c^2.\ \frac{\sin B}{\sin A.\sin C} =$$
$$\frac{ac}{\sin B} + \frac{\sin C}{ab} + \frac{bc}{\sin A}.$$

Hinc autem colligimus, veram esse hanc aequationem:

$$\triangle HIK = \triangle - \tfrac{1}{2}\sin^2 A'\left[(a^2+b^2+c^2)(\operatorname{cotg} A' - \operatorname{cotg} A - \operatorname{cotg} B - \operatorname{cotg} C) + \frac{\sin C}{ab} + \frac{ac}{\sin B} + \frac{\sin A}{bc}\right] \text{ (§. 40.)}$$

§. 42.

Angulis A', B', C', vel A'', B'', C'' ita comparatis, ut transversae idem punctum una transeant, quum area trianguli HIK plane evanescat, hanc existere aequationem, apertum est:

$$\triangle - \tfrac{1}{2}\sin^2 \overset{(4)}{A'}\left[(a^2+b^2+c^2)(\operatorname{cotg} \overset{(4)}{A'} - \operatorname{cotg} A - \operatorname{cotg} B - \operatorname{cotg} C) + \frac{ab}{\sin C} + \frac{ac}{\sin B}\ \frac{bc}{\sin A}\right] = o;$$

inde

$$\triangle = \tfrac{1}{2}\sin^2 \overset{(4)}{A'}\left[(a^2+b^2+c^2)(\operatorname{cotg} \overset{(4)}{A'} - \operatorname{cotg} A - \operatorname{cotg} B - \operatorname{cotg} C) + \frac{ab}{\sin C} + \frac{ac}{\sin B} + \frac{bc}{\sin A}\right],$$

et, quum sit

$$\triangle = \tfrac{1}{2}\left[\frac{ab}{\sin C} + \frac{ac}{\sin B} + \frac{bc}{\sin A}\right]\sin^2 \overset{(4)}{A'} \text{ **)}$$

$$\tfrac{1}{2}\sin^2 \overset{(4)}{A'}\,(a^2+b^2+c^2)(\operatorname{cotg} \overset{(4)}{A'} - \operatorname{cotg} A - \operatorname{cotg} B - \operatorname{cotg} C) = o.$$

Nunquam autem fieri potest, ut sit

$$\text{aut}\quad \sin^2 \overset{(4)}{A'} = o$$
$$\text{aut}\quad a^2+b^2+c^2 = o,$$

**) patet nimirum, esse:

$$\triangle = \triangle AGB + \triangle BGC + \triangle AGC \text{ (Fig. 9.)}$$
$$= \tfrac{1}{2}\sin \overset{(4)}{A'}\,[c.AG + a.BG + b.CG] = \tfrac{1}{2}\sin^2 \overset{(4)}{A'}\left[\frac{ab}{\sin C} + \frac{ac}{\sin B} + \frac{bc}{\sin A}\right],$$

quoniam habemus: $AG = b\ \dfrac{\sin \overset{(4)}{A'}}{\sin A}$; $BG = c.\ \dfrac{\sin \overset{(4)}{A'}}{\sin B}$;

$CG = a.\ \dfrac{\sin \overset{(4)}{A'}}{\sin C}$

itaque vera sit haec aequatio

$$\cot g \overset{(*)}{A'} - \cot g\, A - \cot g\, B - \cot g\, C = o\,;$$

$$\text{vel}\ \cot g\, A' = \cot g\, A + \cot g\, B + \cot g\, C$$

necesse est; iterum igitur hic nobis offertur theorema illud elegantissimum, quod jam supra (§. 22.) alia via demonstravimus.

§. 43.

Quodsi ducuntur in quopiam triangulo (Fig. 17.) praeter quartae ternionis transversas AD', BE', CF', simul et aliae AD, BE, CF, non unum quidem punctum una transeuntes, sed ita comparatae, ut sit $A' = B' = C'$, illos quoque angulos, ad quos altera alteri ternioni occurrit, DAD', EBE', FCF' invicem aequari per se satis apparet. Ex quo trianguli HIK aream tanquam functionem non minus anguli DAD, quam BAD spectari posse, concludamus licet.

Itaque, quia multum commodi nobis inde redundabit, hanc functionem diligentius investigemus.

Quod vero ut facilius fieri possit, pulcherrimi cujusdam theorematis antea mentionem facere expedit. Nimirum si dato quopiam triangulo ABC, (Fig. 19.) aliud HIK huic simile eo, quo antea memoravimus, modo oritur, lineas quartae ternionis transversas in utroque triangulo ductas idem punctum G una transire necesse est.

Transversas enim IL, HM, KN, in triangulo HIK ita ductas, ut punctum G, quo quartae ternionis transversae in triangulo primario ductae invicem secantur, transeant, semper ita comparatas esse, ut sit

$$< HIL = < KHM = < IKN = \overset{(*)}{A'},$$

facile demonstrari potest.

Nam ex ipsa constructione

$$< HAG = < HBG$$

esse sequitur; ergo quadrangulo $ABHG$ circulus circumscribi potest; quare sit

$$< KHM = BAG = \overset{(*)}{A'}$$

similique ratione $$< IKN = < CBG = \overset{(*)}{A'}$$

$$< HIL = < ACG = \overset{(*)}{A'}$$

necesse est [15]).

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[15]) Cl. Crellius, cui et hoc theorema debemus, demonstrationem ejus, a nostra plane diversam, tradidit in l. l. p. 52. sq.

Cor. Quo quidem theoremate alia (§. 21.) nobis ostenditur via, qua infinita multitudo triangulorum, dato cuipiam similium, et uni puncto tanquam communi centro circumjacentium, construi possit.

§. 44.

His demonstratis, quomodo trianguli HIK area ex anguli DAD', quem litera φ significabimus, quantitate pendeat, explorare licet.

In triangulo AGI sit

$$AG : GI :: \sin \overset{(*)}{A'} : \sin \varphi$$

necesse est; inde, quum habeamus

$$AG = b. \frac{\sin A'}{\sin A} \text{ et } GI = KI. \frac{\sin A'}{\sin A},$$

$$AG : GI :: b : KI :: \sin \overset{(*)}{A'} : \sin \varphi,$$

eamque ob rem

$$\triangle : \triangle\ HIK :: b^2 : \overline{KI}^2 :: \sin^2 \overset{(*)}{A'} : \sin^2 \varphi,$$

ergo $$\triangle\ HIK = \frac{\sin^2 \varphi}{\sin^2 \overset{(*)}{A'}} . \triangle;$$

quae quidem aequatio propter miram, qua delectamur, simplicitatem, illi, quam antea (§. 41.) evolvimus, longe est praeferenda.

§. 45.

Jam vero, quae sequantur ex aequatione nostra, videamus.

1) Trianguli HIK area maxima tum efficitur, cum $\sin^2 \varphi$ est maximum, ergo angulo φ explente 90 gradus,
   h. e. omnium triangulorum innumerabilium, et inter se ipsa, et primario similium, maximum est illud, quod fit a transversis, quae quartae ternionis lineas ad angulos rectos secant.

2) Pari modo area trianguli fit minima, cum $\sin^2 \varphi$ est minimum, ergo cum est $\varphi = o$; quod quidem per se clarum atque apertum est; nam tum transversae punctum G una transeunt, eamque ob caussam area trianguli plane evanescit.

3) Nec minus aequatio nostra docet, aream trianguli HIK areae primarii aequalem fieri, si sit

$$\sin^2 \varphi = \sin^2 \overset{(*)}{A'},$$

ergo $\varphi = \overset{(*)}{A'}$.

Per se autem, rem sic se habere, apparet, si transversae ab eadem quartae ternionis linearum parte ducuntur, a qua anguli aequales A', B', C' sunt siti; triangula enim tum se invicem tegant necesse est. Quod quidem fieri nequit, si transversae ab altera parte ductae sunt; attamen area trianguli, nunc ab iis facti, eadem, quae prioris, est; itaque jure hinc colligimus: transversas ita ductas, ut trianguli lateribus ad angulos occurrant, quorum quisque anguli $\overset{(*)}{A'}$ est duplus, triangulum primario congruum efficere.

4) quae transversae trianguli lateribus ad perpendicula normatae insistunt, hanc praebent aequationem:

$$\sin\varphi = \cos\overset{(*)}{A'},$$

inde

$$\triangle\, HIK = \frac{\cos^2\overset{(*)}{A'}}{\sin^2\overset{(*)}{A'}}.\,\triangle = \operatorname{cotg}^2\overset{(*)}{A'}.\,\triangle$$
$$= (\operatorname{cotg} A + \operatorname{cotg} B + \operatorname{cotg} C)^2\,\triangle$$
$$= 2\,\triangle + (\operatorname{cotg}^2 A + \operatorname{cotg}^2 B + \operatorname{cotg}^2 C)\,\triangle$$

5) Triangulum HIK maximum fieri, si sit

$$\varphi = 90^\circ,$$

jam supra diximus; tunc vero aequatio nostra in hanc mutatur:

$$\triangle\, HIK = \frac{1}{\sin^2\overset{(*)}{A'}}\,\triangle = \operatorname{cosec}^2\overset{(*)}{A'}.\,\triangle$$
$$= (1 + \operatorname{cotg}^2\overset{(*)}{A'})\,\triangle$$

h. e. triangulum illud, quod est omnium, quae hoc modo fieri possunt, maximum, areas eorum complectitur, quorum alterum ipsum est primarium, alterum a transversis, hujus latera ad angulos rectos secantibus, ortum est.

6) Quodsi est

$$\varphi = 30^\circ,$$
$$\sin^2\varphi = \tfrac{1}{4}$$

esse constat; ergo

$$\triangle\, GHI = \tfrac{1}{4}\operatorname{cosec}^2\overset{(*)}{A'}.\,\triangle$$

h. e. aream ejus trianguli, quod a transversis, ad angulos 30° graduum quartae ternionis lineis occurrentibus, effectum est, maximum illud triangulum quater complectitur.

7) Quod vero triangulum LMN, circulorum, quorum ope quartae ternionis lineas duximus, centris inter se junctis, oritur, quartam (Fig. 27.) maximi illius efficere partem, demonstrari potest. Quum enim non solum sit angulus $GAK = 90°$, sed etiam puncta G, R in circuitu sint sita (est enim $AGC + AKC = 2\ R$), linea GK centrum M transeat necesse est; habemus igitur:

$$LM : IK :: GM : GK :: 1 : 2$$

et hanc ob caussam

$$\triangle\ HIK = 4\ \triangle\ LMN$$

Itaque triangulo LMN illud, quod fit a transversis, quae quartae ternionis lineas ad angulos 30 graduum secant, aequari videmus.

8) Ponamus esse $\varphi = 45°$;
tum fieri $\sin^2 \varphi = \frac{1}{2}$
goniometria docet; ergo nunc

$$\triangle\ HIK = \tfrac{1}{2} \operatorname{cosec}^2 \overset{(4)}{A'}.\ \triangle$$

h. e. transversae eae, quae quartae ternionis lineas ad angulos 45 graduum transeunt, triangulum, cujus area dimidiam maximi illius efficit partem, constituunt.

9) Tribuatur jam angulo φ ea magnitudo, qua sit $= \frac{2}{3} R$; patet nunc esse

$$\sin^2 \varphi = \tfrac{3}{4}$$

et hanc ob rem

$$\triangle\ GHI = \tfrac{3}{4} \operatorname{cosec}^2 \overset{(4)}{A'}.\ \triangle$$

h. e. si transversae ita ducuntur, ut anguli, ad quos quartae ternionis transversas secant, sint 60 graduum, triangulum, quod existit, tres quartas areae maximi illius continet.

10) Triangulum illud HIK, (Fig. 10.) quod perpendiculis a puncto G in latera primarii demissis eorumque finibus inter se junctis oritur, esse

$$= \frac{\triangle}{\operatorname{cosec}^2 \overset{(4)}{A'}}$$

quum jam supra (§. 21.) demonstratum sit, hinc primarium esse geometricae proportionis medium inter hoc et maximum illud colligimus.

Quibus quidem diversis triangulis similibus ita significatis, ut signo $\triangle$ numerum adponamus, secundum quem triangulum ipsum modo percensuimus, hae nobis aequationes offeruntur.

1) $\overset{(3)}{\triangle} = \triangle$

2) $\overset{(1)}{\triangle} = \overset{(4)}{\triangle} + \triangle = \overset{(4)}{\triangle} + \overset{(5)}{\triangle}$

3) $\overset{(1)}{\triangle} = 4\overset{(6)}{\triangle} = 4\overset{(7)}{\triangle} = \overset{(8)}{\triangle} + \overset{(9)}{\triangle}$

$$4)\ \overset{(4)}{\Delta} = 2\overset{(3)}{\Delta} = \overset{(6)}{\Delta} + \overset{(7)}{\Delta} = \overset{(7)}{\Delta} + \overset{(9)}{\Delta}$$

$$5)\ \Delta^{2} = \overset{(1)}{\Delta}.\overset{(10)}{\Delta}$$

Scholion. Omnibus iis, quae modo traditae sunt, proprietatibus, quartam et quintam ternionem convenire, tam facile perspicitur, ut explicandae huic rei immorari non opus sit.

§. 46.

Quemadmodum autem triangula, ab iis transversis facta, quarum quaevis uni alterive laterum, angulum, a cujus vertice ducta est, intercipientium, ad ejusdem magnitudinis angulum occurrit, omnia primario sunt similia, ita idem fiat necesse est, si transversae latera opposita (vel ipsa vel producta) ad ejusmodi angulos transeunt. Nam triangulum v. c. HIK (Fig. 20.) primario ABC simile esse, si sint anguli ADC, BEA et CFB inter se aequales, facile intelligitur; nec minus, trianguli HIK aream ex quantitate anguli ADC pendere, patet.

Sed nunc quoque disquisitioni, de hujus functionis forma instituendae, haud parum expedit, non ipsum angulum ADC, sed eum DAD', ad quem eadem transversa perpendiculo AD' occurrit, quemque in sequentibus litera φ' significabimus, tanquam eam quantitatem spectari, quae areae trianguli HIK determinandae inserviat.

Quod quidem fieri omnino posse jam ex eo elucet, quod angulus φ' anguli ADC est complementum, et hanc ob caussam certa quantitate huic attributa, illius magnitudo arbitraria esse nequit.

§. 47.

Priusquam vero rem ipsam aggrediamur, aliam propositionem, quae in exploranda functionis nostrae forma subsidio nobis erit, proferre liceat. Omnia nimirum illa triangula, et inter se, et primario similia, quae eo, quo diximus, modo existunt, ita sunt comparata, ut punctum G, in quo perpendicula, a verticibus in latera primarii demissa, sese transeunt, circuli, singulo cuique circumscribendi, sit centrum.

Demissis enim perpendiculis GL, GM, GN, ductisque lineis GH, GI, GK, LN, quadrangulis $AGCI$, $BF'GL$, et $HNGL$ circumscribi posse circulos patet; sit ergo necesse est:

$$\angle AIG = \angle ACG = \angle ABG = \angle NLG = \angle NHG$$

et hanc ob caussam

$$\Delta\, GNH \cong \Delta\, GNI;$$

ergo $GH = GI$, similique ratione

$$GK = GI;$$

quare G est centrum circuli triangulo HIK circumscribendi.

Propositionem vero nostram, cujuscunque sit magnitudinis angulus φ', valere per se satis adparet.

§. 48.

Jam investigare licet, quomodo trianguli HIK area anguli φ' magnitudini sit obnoxia.

Nunquam fieri potest (Fig. 20.), quin sit

$$AG : GI :: \sin AIG : \sin \varphi'$$
$$AG : c :: \cos A : \sin C$$

et, quoniam GI radium circuli, qui triangulo HIK circumscribi potest, esse cognovimus,

$$GI = \frac{KI}{2 \sin C}$$
$$\angle AIG = \tfrac{1}{2}(-A + B + C),$$

ergo

$$c. \frac{\cos A}{\sin C} : \frac{KI}{2 \sin C} :: \sin \tfrac{1}{2}(-A + B + C) : \sin \varphi'$$
$$c : KI :: \frac{\sin \tfrac{1}{2}(-A+B+C)}{\cos A} : 2 \sin \varphi'$$
$$c^2 : \overline{KI}^2 :: \left(\frac{\sin \tfrac{1}{2}(-A+B+C)}{\cos A}\right)^2 : 4 \sin^2 \varphi'.$$

Est autem $\tfrac{1}{2}(-A+B+C) + A = \tfrac{1}{2}(A+B+C) = 90°$,

ergo $\sin \tfrac{1}{2}(-A+B+C) = \cos A,$

unde

$$\triangle : \triangle\ HIK :: c^2 : \overline{KI}^2 :: 1 : 4 \sin^2 \varphi'$$
$$\text{vel} \quad \triangle\ HIK = 4 \sin^2 \varphi' \cdot \triangle;$$

simplicissimam ergo functionis nostrae esse formam videmus.

§. 49.

Quae quidem aequatio haec theoremata nobis prabet:

1) Area trianguli HIK fit maxima, si $\sin^2 \varphi'$ habemus maximum, ergo si $\varphi' = 90°$; h. e. omnium triangulorum, quae hoc modo existere possunt, maximum est illud, cujus latera tertiae ternionis lineas ad angulos rectos secant, vel quod idem est, lateribus primarii oppositis sunt parallela.

2) Ipsum hoc triangulum maximum primarii aream quater complectitur; est enim nunc $\sin^2 \varphi' = 1$, et hanc ob caussam

$$\triangle\ HIK = 4\ \triangle$$

3) Eodem modo area trianguli HIK et $\sin^2 \varphi'$ una efficiuntur minima; ergo si est

$$\varphi' = 0$$

quod quidem satis apertum. Scimus enim perpendicula a verticibus in latera demissa in uno puncto sese transire.

4) Jam ponamus, esse

$$\varphi' = 30°;$$

quoniam autem tum

$$4 \sin^2 \varphi' = 1$$

esse constat, hanc aequationem nanciscimur:

$$\triangle\, HIK = \triangle$$

h. e. quae transversae ita ducuntur, ut tertiae ternionis lineas ad angulos 30 graduum, vel quod idem valet, latera primarii opposita ad angulos 60 graduum transeant, triangulum primario congruum efficinut.

5) Quodsi esse

$$\varphi' = 45°$$

sumitur, est $\sin^2 \varphi = 2$

$$\text{ergo} \quad \triangle\, HIK = 2\, \triangle$$

h. e. si transversae ita ducuntur, ut tertiae ternionis lineis, vel lateribus primarii oppositis ad angulos 45 graduum occurrant, triangulum ab iis factum, duplum primarii aequat.

6) Jam sit

$$\varphi' = 60°;$$

constat autem, tum esse

$$4 \sin^2 \varphi' = 3$$

eamque ob caussam

$$\triangle\, HIK = 3\, \triangle$$

h. e. ut triangulum, quod primarii sit triplum, existat, lineae transversae ternionis tertiae lineas ad angulos 60 graduum, vel, quod idem est, latera primarii opposita ad angulos 30 graduum secent necesse est.

7) Quodsi angulum φ' angulo BAD' vel $\overset{(2)}{A'}$ aequari ponimus, sequitur, esse

$$\sin^2 \varphi' = \cos^2 B$$

$$\text{et} \quad \triangle\, HIK = 4 \cos^2 B . \triangle$$

Sunt autem triangula ABC, et $BD'F'$ inter se similia,

ergo

$$\triangle\, BD'F' : \triangle :: \overline{BD'}^2 : c^2$$
$$:: c^2 . \cos^2 B : c^2$$
$$:: \cos^2 B : 1,$$

unde colligimus, nunc esse

$$\triangle\, HIK = 4\, \triangle\, BD'F'$$

Simili plane modo nanciscimur,

8) si est $\varphi' = \overset{(8)}{B'}$

$$\triangle\, HIK = 4 \triangle\, CD'E'$$

et

9) si $\varphi' = \overset{(9)}{C'}$

esse sumitur,

$$\triangle\, HIK = 4 \triangle\, AE'F'.$$

10) Itaque si in triangulis $BD'F'$, $CD'E'$, $AE'F'$ transversae eae ducuntur, quae lateribus oppositis sunt parallelae, triangula ab iis facta congrua sunt triangulis, (si singulum singulo compares), quae efficiuntur a tribus transversarum ternionibus, in triangulo primario ita ductis, ut sit primum $\sin\varphi' = \cos B$, deinde $\sin\varphi' = \cos C$, denique $\sin\varphi' = \cos A$.

Si nunc brevitatis caussa singula haec triangula et inter se, et primario similia signo $\triangle$ ita exprimuntur, ut huic supponatur is numerus, secundum quem ipsum, quod innuimus, triangulum, percensuimus, praeterea autem

$$\triangle\, BD'F' = \triangle', \; \triangle\, CD'E' = \triangle'', \text{ et } \triangle\, AE'F' = \triangle'''$$

esse ponitur, hae nobis aequationes prodeunt:

1) $\underset{(1)}{\triangle} = 4 \triangle = 4 \underset{(4)}{\triangle}$

2) $\underset{(4)}{\triangle} = \triangle$

3) $\underset{(5)}{\triangle} = 2 \triangle = 2 \underset{(4)}{\triangle} = \frac{1}{2} \underset{(1)}{\triangle}$

4) $\underset{(6)}{\triangle} = 3 \triangle = 3 \underset{(4)}{\triangle} = \frac{3}{4} \underset{(1)}{\triangle}$

5) $\underset{(4)}{\triangle} + \underset{(6)}{\triangle} = 2 \underset{(5)}{\triangle} = \underset{(1)}{\triangle}$

6) $\underset{(1)}{\triangle'} = \underset{(7)}{\triangle}$

7) $\underset{(1)}{\triangle''} = \underset{(8)}{\triangle}$

8) $\underset{(1)}{\triangle'''} = \underset{(9)}{\triangle}.$

Sectio commentationis tertia.

De circulorum, quibus cum exteriori et interiori trianguli circulo necessitudo quaedam est, proprietatibus.

§. 50.

Circulis illis, quorum ope quartae et quintae ternionis transversas duximus, nec minus aliis quibusdam, prope ad hos accedentibus, plures, praesertim si eos cum circulo et exteriori et interiori compares, quum sint proprietates, omnino, ut nobis videtur, dignae, quae memorentur, breviter nunc eas exponere placet. Brevitatis autem caussa horum quoque diversae distinguendae sunt nobis terniones. Primam et secundam ipsos illos circulos complecti volumus, qui supra ad ducendas quartae et quintae ternionis transversas erant nobis subsidio.

Cujusque ternionis radios literis R', R'', R''' significabimus, ita quidem, ut R' ejus circuli sit radius, qui fines B et C primi lateris a transit etc.

§. 51.

Ex ipsa circulorum constructione, esse (Fig. 21.)

$$\tfrac{1}{2}\, a = R' \cos HCL,$$

sequitur; inde

$$2\, \overset{(1)}{R'} = \frac{a}{\sin C}$$

et simili ratione

$$2\, \overset{(1)}{R''} = \frac{b}{\sin A}$$

$$2\, \overset{(1)}{R'''} = \frac{c}{\sin B},$$

ergo

$$8\, R'.\, \overset{(1)}{R''}.\, R''' = \frac{a.\, b.\, c}{\sin A.\, \sin B.\, \sin C} = \left(\frac{a}{\sin A}\right)^3 = 8\, R^3,$$

unde

$$R'.\, \overset{(1)}{R''}.\, R''' = R^3$$

h. e. parallelepidum rectum, a primae ternionis radiis factum, radii majoris [10]) cubum aequat.

§. 52.

Item ex secundae ternionis circulorum constructione habemus (Fig. 22.):

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[10]) Brevitatis gratia in sequentibus radium circuli exterioris radium majorem adpellabimus, interioris autem radium minorem.

$$\tfrac{1}{2}\, a = \overset{(1)}{R'}.\cos HBL,$$

unde $$2\,\overset{(1)}{R'} = \frac{a}{\sin B}$$

eademque ratione

$$2\,\overset{(1)}{R''} = \frac{b}{\sin C}$$

et $$2\,\overset{(2)}{R'''} = \frac{c}{\sin A},$$

ergo $$8\,\overset{(2)}{R'.R''.R'''} = \frac{a.\,b.\,c}{\sin A.\,\sin B.\,\sin C} = 8\,R^3$$

et $$\overset{(2)}{R'.R''.R'''} = R^3$$

h. e. parallelepipedum rectum, a secundae ternionis radiis factum, radii majoris cubum aequat.

Cor. Facile hinc colligimus, esse

$$\overset{(1)}{R'.R''.R'''} = \overset{(2)}{R'.R''.R'''}$$

h. e. parallelepipeda a primae et secundae ternionis radiis facta inter se sunt aequalia.

§. 53.

Lineis HL, IM, KN ita productis, ut centrum O circuli exterioris transeant, ductisque radiis AO, BO, CO, existunt triangula AOK, BOH, COI, non solum inter se ipsa sed etiam primario similia; locum igitur habeant necesse est hae proportiones:

$$R : HO :: b : c$$
$$R : IO :: c : a$$
$$R : KO :: a : b,$$

ergo $$R^3 = HO.\,IO.\,KO = \overset{(1)}{R'.R''.R'''} = \overset{(2)}{R'.R''.R'''}$$

h. e. parallelepipedum rectum a lineis factum, quae singulae distantiam inter exterioris circuli centrum et primae ternionis centra indicant, radii majoris cubum aequat.

Caroll. Nil autem referre patet, utram ternionem adhibeamus; itaque si lineae HO, IO, KO literis d', d'', d''' significantur, habemus

$$\overset{(1)}{d'.d''.d'''} = \overset{(2)}{d'.d''.d'''} = \overset{(1)}{R'.R''.R'''} = \overset{(2)}{R'.R''.R'''} = R^3$$

§. 54.

Facile perspicitur, esse (Fig. 21.)

$$OL = R.\cos A \quad \text{et} \quad HL = \overset{(1)}{R'}.\cos C$$
$$OM = R.\cos B \qquad IM = \overset{(1)}{R''}.\cos A$$
$$ON = R.\cos C \qquad KN = \overset{(1)}{R'''}.\cos B,$$

ergo

$$OL.OM.ON = R^3.\cos A.\cos B.\cos C = R'.R''.R'''.\cos A.\cos B.\cos C = HL.IM.KN$$

h. e. parallelepipeda recta facta a lineis, quae indicant, quantum a trianguli lateribus distent et primae vel secundae ternionis centra (singulum a singulo, quod hujus circuli est chorda), et centrum circuli exterioris, inter se sunt aequalia.

§. 55.

Primae ternionis radii, in unam summam collecti, hanc nobis aequationem praebent:

$$R' + \overset{(1)}{R''} + R''' = \frac{a}{2\sin C} + \frac{b}{2\sin A} + \frac{c}{2\sin B}$$
$$= \frac{a^2\, b + b^2\, c + c^2\, a}{4\,\triangle},$$

inde

$$(R' + \overset{(1)}{R''} + R''')\, 4\,\triangle = a^2\, b + b^2\, c + c^2\, a$$

h. e. tria illa parallelepipeda recta, ita a trianguli lateribus facta, ut unius eorum quadratum singulo cuique sit basis, alterum vero idque posterius altitudo, prisma id aequant, cujus basis aream trianguli quater complectitur, altitudo autem primae ternionis radiorum summam aequat.

§. 56.

Pari modo habemus:

$$R' + \overset{(2)}{R''} + R''' = \frac{a}{2\sin B} + \frac{b}{2\sin C} + \frac{c}{2\sin A}$$
$$= \frac{a^2\, c + b^2\, a + c^2\, b}{4\,\triangle},$$

unde

$$(R' + \overset{(2)}{R''} + R''')\,\triangle = a^2\, c + b^2\, a + c^2\, b$$

§. 57.

Ex iis, quae modo tradidimus, quoniam est

$$R = \frac{abc}{4\,\triangle},$$

esse sequitur:

$$(3\,R + R' + \overset{(1)}{R''} + R''' + R' + \overset{(2)}{R''} + R''')\,4\,\triangle = 3\,abc + a^2 b + b^2 c + c^2 a + a^2 c + b^2 a + c^2 b$$
$$= (ab + ac + bc)\,(a + b + c)$$

h. e. prisma, cujus basis triangulum est ipsum, altitudo vero summam primae et secundae ternionis radiorum summam, radii majoris triplo auctam, quater complectitur, aequale est parallelepipedo ei, cujus basis rectangulorum, a binis trianguli lateribus factorum, summam, altitudo ejusdem pirimetrum aequat.

Cor. Quum in omni triangulo sit

$$4\,\triangle = 2\,r\,(a + b + c),$$

et $$ab + ac + bc = r.\,r' + r.\,r'' + r.\,r''' + r'.\,r'' + r'.\,r''' + r''.\,r''',$$

colligimus esse:

$$(3\,\mathrm{R} + \mathrm{R}' + \overset{(1)}{\mathrm{R}''} + \mathrm{R}''' + \mathrm{R}' + \overset{(2)}{\mathrm{R}''} + \mathrm{R}''')\,2r = r.r' + r.r'' + r.r''' + r'.r'' + r'.r''' + r''.r'''$$

h. e. rectangulum a primae et secundae ternionis radiorum summa, radii majoris triplo aucta, et diametro minori factum, summam rectangulorum, quae a binis oriuntur radiis circulorum, latera trianguli tangentium, aequat.

§. 58.

Quum demonstratum sit a cl. Feuerbachio (§. 22.), esse:

$$\mathrm{R}.\,(DE + \overset{(2)}{DF} + EF) = r\,(a+b+c),$$

et supra (§. 19.) a nobis:

$$2\,\mathrm{R}\,(d + \overset{(2)}{e} + f) = ab + ac + bc,$$

aequationem, quae antea (§. 57.) legitur, in hanc mutari posse videmus:

$$(3\,\mathrm{R} + \mathrm{R}' + \overset{(1)}{\mathrm{R}''} + \mathrm{R}''' + \mathrm{R}' + \overset{(2)}{\mathrm{R}''} + \mathrm{R}''')\,(DE + \overset{(2)}{DF} + EF) = (a+b+c)\,(d + \overset{(2)}{e} + f)$$

h. e. rectangulum a primae secundaeque ternionis radiorum summa, radii majoris triplo aucta, et linearum, quae perpendiculorum fines jungunt, summa factum, rectangulo aequale est ei, quod ab ipsorum perpendiculorum summa trianguli que perimetro ortum est.

§. 59.

Binis primae ternionis radiis invicem multiplicatis, habemus:

$$\overset{(1)}{\mathrm{R}'}.\,\mathrm{R}'' = \frac{a.\,b}{4\sin A.\sin C};\quad \overset{(1)}{\mathrm{R}'}.\,\mathrm{R}''' = \frac{a.\,c}{4\sin B.\sin C};\quad \overset{(1)}{\mathrm{R}''}.\,\mathrm{R}''' = \frac{b.\,c}{4\sin A.\sin B},$$

inde

$$R'.R''+\overset{(1)}{R'}.R'''+R''.R''' = \frac{a}{2\sin A}\left[\frac{b}{2\sin C}+\frac{a}{2\sin B}+\frac{c}{2\sin A}\right]$$
$$= R\ [R'+\overset{(2)}{R''}+R''']$$

h. e. rectangulorum, a binis primae ternionis radiis factorum, summa rectangulum id aequat, quod a radio majori et secundae ternionis radiorum summa oritur.

§. 60.

Simili ratione esse videmus:

$$R'.R''+\overset{(2)}{R'}.R'''+R''.R''' = \frac{a\,b}{4\sin B\sin C}+\frac{a\,c}{4\sin A.\sin B}+\frac{b\,c}{4\sin A.\sin C}$$
$$= \frac{b}{2\sin B}\left[\frac{a}{2\sin C}+\frac{c}{2\sin B}+\frac{b}{2\sin A}\right]$$
$$= R.\ [R'+\overset{(1)}{R''}+R''']$$

§. 61.

Ex iis, quae modo demonstravimus, esse

$$\frac{R'.R''+\overset{(1)}{R'}.R'''+R''.R'''}{R'+\overset{(2)}{R''}+R'''} = \frac{R'.R''+\overset{(2)}{R'}.R'''+R''.R'''}{R'+\overset{(1)}{R''}+R'''} = R,$$

facile sequitur; unde

$$(R'.R''+\overset{(1)}{R'}.R'''+R''.R''')(R'+\overset{(1)}{R''}+R''') = (R'.R''+\overset{(2)}{R'}.R'''+R''.R''')(R'+\overset{(2)}{R''}+R''')$$

h. e. quod parallelepipedum ita construitur, ut basis rectangulorum, a binis primae ternionis radiis factorum, summam, altitudo autem ejusdem ternionis radiorum summam aequet, id alteri parallelepipedo, eodem plane modo a secundae ternionis radiis orto, aequatur.

Schol. Propositionis nostrae formam mutari posse in hanc:

$$(R'+\overset{(1)}{R''}+R''')\left(\frac{1}{R'}+\overset{(1)}{\frac{1}{R''}}+\frac{1}{R'''}\right) = (R'+\overset{(2)}{R''}+R''')\left(\frac{1}{R'}+\overset{(2)}{\frac{1}{R''}}+\frac{1}{R'''}\right),$$

facile perspicitur.

§. 62.

Quodsi rectangula, ab iis duarum ternionum radiis facta, quorum circuli eidem trianguli lateri, tanquam chordae circumscripti sunt, in unam summam coguntur, haec aequatio prodeat necesse est:

$$\overset{(1)}{\mathrm{R}}'.\overset{(2)}{\mathrm{R}}' + \overset{(1)}{\mathrm{R}}''.\overset{(2)}{\mathrm{R}}'' + \overset{(1)}{\mathrm{R}}'''.\overset{(2)}{\mathrm{R}}''' = \frac{a^2}{4\sin B.\sin C} + \frac{b^2}{4\sin A.\sin C} + \frac{c^2}{4\sin A.\sin B}$$

$$= \frac{a^3bc + ab^3c + abc^3}{16\,\triangle^2}$$

$$= \frac{abc}{4\triangle} \cdot \frac{a^2 + b^2 + c^2}{4\triangle},$$

unde

$$\frac{\mathrm{R}'.\mathrm{R}' + \overset{(1)}{\mathrm{R}}''.\overset{(2)}{\mathrm{R}}'' + \mathrm{R}'''.\mathrm{R}'''}{\mathrm{R}}.\,4\triangle = a^2 + b^2 = c^2$$

h. e. rectangulum, eam, quam modo diximus, summam rectangulorum quater complectens, cujus alterum latus est radius major, ita semper comparatum est, ut ejus alterum latus altitudinem prismatis aequet, cujus basis est triangulum ipsum, quodque cuborum, super singulis ejusdem lateribus constructorum, summae aequatur.

§. 63.

Facili negotio nunc esse intelligitur:

$$(a+b+c)^2 = \left[3(2\mathrm{R} + \mathrm{R}' + \overset{(1)}{\mathrm{R}}'' + \mathrm{R}''' + \mathrm{R}' + \overset{(2)}{\mathrm{R}}'' + \mathrm{R}''') + \frac{\mathrm{R}'.\mathrm{R}' + \overset{(1)}{\mathrm{R}}''.\overset{(2)}{\mathrm{R}}'' + \mathrm{R}'''\mathrm{R}'''}{\mathrm{R}}\right] 4\triangle;$$

inde

$$\mathrm{R}\,(a+b+c)^2 = [3\,\mathrm{R}\,(2\,\mathrm{R} + \mathrm{R}' + \overset{(1)}{\mathrm{R}}'' + \mathrm{R}''' + \mathrm{R}' + \overset{(2)}{\mathrm{R}}'' + \mathrm{R}''') + \mathrm{R}'.\mathrm{R}' + \overset{(1)}{\mathrm{R}}''.\overset{(2)}{\mathrm{R}}'' + \mathrm{R}'''.\mathrm{R}''']\,.\,4\triangle;$$

ergo, quum sit $4\triangle = 2r\,(a+b+c)$,

$$\mathrm{R}(a+b+c)^2 = [3\mathrm{R}\,(2\,\mathrm{R} + \mathrm{R}' + \overset{(1)}{\mathrm{R}}'' + \mathrm{R}''' + \mathrm{R}' + \overset{(2)}{\mathrm{R}}'' + \mathrm{R}''') + \mathrm{R}'.\mathrm{R}' + \overset{(1)}{\mathrm{R}}''.\overset{(2)}{\mathrm{R}}'' + \mathrm{R}'''.\mathrm{R}''']\;2r$$

Cor. Quum cl. Feuerbachius (§. 6.)

$$r'.r'' + r'.r''' + r''.r''' = \tfrac{1}{4}\,(a+b+c)^2$$

esse demonstraverit, haec quoque aequatio

$$[3\mathrm{R}\,(2\mathrm{R} + \mathrm{R}' + \overset{(1)}{\mathrm{R}}'' + \mathrm{R}''' + \mathrm{R}' + \overset{(2)}{\mathrm{R}}'' + \mathrm{R}''') + \mathrm{R}'.\mathrm{R}' + \overset{(1)}{\mathrm{R}}''.\overset{(2)}{\mathrm{R}}'' + \mathrm{R}'''.\mathrm{R}''']\;r = (r'.r'' + r'.r''' + r''.r''')\;2\,\mathrm{R}$$

vera sit necesse est.

§. 64.

Ex iis, quae supra (§. 55. et §. 56.) leguntur, facile colligimus, esse:

F 2

$$R+R'+\overset{(1)}{R}''+R'''+R+R'+\overset{(2)}{R}''+R''' = \frac{2abc+a^2b+b^2c+c^2a+a^2c+b^2a+c^2b}{4\,\triangle}$$
$$= \frac{a(b+c)^2+b(c+a)^2+c(a+b)^2-4abc}{4\,\triangle};$$

inde

$$(3R+R'+\overset{(1)}{R}''+R'''+3R+R'+\overset{(2)}{R}''+R''').\,4\triangle = a(b+c)^2+b(c+a)^2+c(a+b)^2$$

h. c. prisma illud, cujus basis est triangulum ipsum, altitudo autem primae et secundae ternionis radiorum summam, sexduplo radii majoris auctam, complectitur, aequale est summae trium parallelepipedorum, quae singula a duorum trianguli laterum summae quadrato tanquam basi, et tertio ejusdem latere tanquam altitudine fiunt.

§. 65.

Per se satis adparet, esse:

$$\overset{(2)}{R}'.\overset{(1)}{R}'' = \frac{a}{2\sin B}\cdot\frac{b}{2\sin A} = \frac{a^2}{4\sin^2 A};$$
$$\overset{(1)}{R}'.\overset{(2)}{R}''' = \frac{a}{2\sin C}\cdot\frac{c}{2\sin A} = \frac{a^2}{4\sin^2 A};$$
$$\overset{(2)}{R}''.\overset{(1)}{R}''' = \frac{b}{2\sin C}\cdot\frac{c}{2\sin B} = \frac{a^2}{4\sin^2 A};$$

ergo $\overset{(2)}{R}'.\overset{(1)}{R}'' = \overset{(1)}{R}'.\overset{(2)}{R}''' = \overset{(2)}{R}''.\overset{(1)}{R}''' = R^2$

h. c. rectangula, ab iis duarum ternionum radiis, quorum circuli idem latus trianguli tangunt, facta, radii majoris quadratum aequant.

§. 66.

Quum fieri nequeat, quin verae sint aequationes hae:

$$R'^2+\overset{(1)}{R}''^2+R'''^2 = (R'+\overset{(1)}{R}''+R''')^2 - 2(R'.R''+\overset{(1)}{R}'.R'''+R''.R'''),$$

et $$R'^2+\overset{(2)}{R}''^2+R'''^2 = (R'+\overset{(2)}{R}''+R''')^2 - 2(R'R''+\overset{(2)}{R}'.R'''+R''.R'''),$$

concludamus licet, esse.

$$R'^2+\overset{(1)}{R}''^2+R'''^2+R'^2+\overset{(2)}{R}''^2+R'''^2 = (R'+\overset{(1)}{R}''+R''')^2+(R'+\overset{(2)}{R}''+R''')^2-2R(R'+\overset{(1)}{R}''+R'''$$
$$-2R(R'+\overset{(2)}{R}''+R''') \quad (\S.59.\text{ et }\S.60.);$$

inde

$$R^2+R'^2+\overset{(1)}{R}{}''^2+R'''^2+R^2+R'^2+\overset{(2)}{R}{}''^2+R'''^2=(R'+\overset{(1)}{R}{}''+R'''-R)^2+(R'+\overset{(2)}{R}{}''+R'''-R)^2$$

h. e. quadratorum, ab utriusque ternionis radiis factorum, summa, quadrati, a majori radio orti, duplo aucta, summam quadratorum eorum aequat, quae super lineis constructa sunt, quantum et primae et secundae ternionis radiorum summa radium majorem excedat, indicantibus.

§. 67.

Patet omnino esse:

$$a.\frac{b}{\sin C}=b.\frac{a}{\sin C};\ a.\frac{c}{\sin B}=c.\frac{a}{\sin B};\ b.\frac{c}{\sin A}=c.\frac{b}{\sin A};$$

ergo

$$a\overset{(2)}{R}{}''=b.\overset{(1)}{R}{}';\ b.\overset{(2)}{R}{}'''=c.\overset{(1)}{R}{}'';\ c.\overset{(2)}{R}{}'=a\overset{(1)}{R}{}'''$$

h. e. bina rectangula ea, quorum alterum a quolibet trianguli latere et secundae ternionis radio tangente [20]), alterum vero ab ejusdem latere sequenti et primae ternionis radio tangente factum est, inter se sunt aequalia.

§. 68.

Non minus apertum est, locum omnino habere aequationem hanc:

$$\frac{a}{\sin A}\cdot bc+\frac{b}{\sin B}\cdot ac+\frac{c}{\sin C}\cdot ab=\frac{a}{\sin C}\cdot bc+\frac{b}{\sin A}\cdot ac+\frac{c}{\sin B}\cdot ab=\frac{a}{\sin B}\cdot bc+\frac{b}{\sin C}\cdot ac+\frac{c}{\sin A}\cdot ab$$

ergo

$$(ab+ac+bc)\,R=ab\overset{(1)}{R}{}'''+ac\overset{(1)}{R}{}''+bc\overset{(1)}{R}{}'=ab\overset{(2)}{R}{}'''+ac\overset{(2)}{R}{}''+bc\overset{(2)}{R}{}'$$

h. e. parallelepipedorum rectorum, quae singula a singulis primae vel secundae ternionis radiis et iis trianguli lateribus fiunt, quorum neutrum hujus circuli chorda est, summa parallelepipedo aequalis est ei, cujus basis rectangulorum, a binis trianguli lateribus ortorum, summam, altitudo autem radium majorem aequat.

§. 69.

Per se satis apparet, esse:

[20]) radio tangente brevitatis causa dictum puta pro radio circuli tangentis.

$$\frac{a}{\sin A}\cdot\frac{b}{\sin A}\cdot c+\frac{b}{\sin B}\cdot\frac{c}{\sin B}\cdot a+\frac{c}{\sin C}\cdot\frac{a}{\sin C}\cdot b=\frac{a}{\sin A}\cdot\frac{c}{\sin A}\cdot b+\frac{b}{\sin B}\cdot\frac{a}{\sin B}\cdot c+\frac{c}{\sin C}\cdot\frac{b}{\sin C}\cdot a=a.\frac{b}{\sin A}\cdot\frac{c}{\sin A}+b.\frac{a}{\sin B}\cdot\frac{c}{\sin B}+c.\frac{a}{\sin C}\cdot\frac{b}{\sin C}$$

et hanc ob caussam:

$$R\left[a\overset{(1)}{R}{}'''+b\overset{(1)}{R}{}'+c.\overset{(1)}{R}{}''\right]=R\left[a\overset{(2)}{R}{}''+b\overset{(2)}{R}{}'''+c\overset{(2)}{R}{}'\right]=a\overset{(1)}{R}{}''.\overset{(2)}{R}{}'''+b.\overset{(2)}{R}{}'.\overset{(1)}{R}{}'''+c.\overset{(1)}{R}{}'.\overset{(2)}{R}{}''$$

h. e. parallelepipeda recta, a binis radiis, quorum circuli in eodem puncto duo trianguli latera tangunt, et a tertio ejusdem latere facta, si in unam summam coguntur, parallelepipedum illud aequant, cujus basis trium rectangulorum, quae singula a singulis trianguli lateribus et ab alterutrius ternionis circulorum, ipsa haec latera tangentium, radiis oriuntur, aream continet, cujusque altitudo radius est major.

§. 70.

Extra omnem positum est dubitationem, esse:

$$\frac{a}{\sin A}\cdot\frac{b}{\sin B}\cdot c+\frac{a}{\sin A}\cdot\frac{c}{\sin C}\cdot b+\frac{b}{\sin B}\cdot\frac{c}{\sin C}\cdot a=\frac{a}{\sin B}\cdot\frac{c}{\sin A}\cdot b+\frac{b}{\sin C}\cdot\frac{c}{\sin A}\cdot a+$$
$$+\frac{a}{\sin B}\cdot\frac{b}{\sin C}\cdot c=\frac{b}{\sin A}\cdot\frac{c}{\sin B}\cdot a+\frac{a}{\sin C}\cdot\frac{b}{\sin A}\cdot c+\frac{a}{\sin C}\cdot\frac{c}{\sin B}\cdot b\,;$$

unde

$$R^2(a+b+c)=a.\overset{(2)}{R}{}''.R'''+b.\overset{(2)}{R}{}'.R'''+c\overset{(2)}{R}{}'.R''=a\overset{(1)}{R}{}''.R'''+b.\overset{(1)}{R}{}'.R'''+c.\overset{(1)}{R}{}'.R''$$

h. e. parallelepipedum rectum, a radii majoris quadrato et trianguli perimetro factum, summam parallelepipedorum rectorum aequat, quae singula a duobus ejusdem ternionis radiis et eo trianguli latere, quod neutrius horum circulorum est chorda, oriuntur.

§. 71.

Quisque sponte intelligit, esse:

$$\frac{a}{\sin A}\cdot b+\frac{c}{\sin C}\cdot a+\frac{b}{\sin B}\cdot c.=a.\frac{b}{\sin A}+c.\frac{a}{\sin C}+b.\frac{c}{\sin B},$$

ergo

$$R\,(a+b+c)=a\overset{(1)}{R}{}''+b\overset{(1)}{R}{}'''+c.\overset{(1)}{R}{}'$$

h. e. rectangula ea, quorum singulum quodque fit a singulo primae ternionis radio et latere trianguli illo, quod neque tangens neque chorda est hujus circuli,

in unam summam collecta, rectangulo, a radio majori perimetroque trianguli orto, aequatur.

§. 72.

Ex iis, quae cl. Feuerbachius (§. 5.) demonstravit, in omni triangulo esse

$$r' + r'' + r''' = r + 4\,R$$

constat; ergo est

$$(R' + \overset{(1)}{R''} + R''')\,(r' + r'' + r''') = (R' + \overset{(1)}{R''} + R''')\,(r + 4\,R),$$

unde

$$(R' + \overset{(1)}{R''} + R''')\,(r' + r'' + r''' - r) = 4\,R\,(R' + \overset{(1)}{R''} + R''')$$

$$= 4\,(R'.\,R'' + R'.\overset{(1)}{R'''} + R''.\,R''')$$

h. e. rectangulum, a primae ternionis radiorum summa et linea ea factum, qua radiorum, quorum circuli latera trianguli extrinsecus tangunt, summa radium minorem superat, summam rectangulorum, a binis secundae ternionis radiis ortorum, quater complectitur.

Cor. Eadem via, esse

$$(R' + \overset{(1)}{R''} + R''')\,(r' + r'' + r''' - r) = 4\,(R'.\,R'' + R'.\overset{(1)}{R'''} + R''.\,R''')$$

demonstratur.

§. 73.

Ipsa ea, quae hucusque de primae et secundae ternionis circulis tradidimus, prope quidem eos ad circulum interiorem, sed multo propius ad exteriorem accedere, luculenter docent. Transeamus jam ad illos, quibus inversa ad utrumque circulum est ratio. Facile autem intelligitur, referendos huc esse circulos eos, qui bina trianguli latera ita tangunt, ut tertii lateris alterum finem transeant. Itaque duas iterum horum circulorum terniones esse videmus; alterius nimirum circuli lateris tertii finem priorem, alterius posteriorem transeunt; huic ternionis tertiae, illi quartae sit nomen.

Quaenam constructio ad utriusque ternionis circulos describendos adhibenda sit, tam facile est intellectu, ut omni explicatione supersedere possimus.

Ex ipsa autem hac constructione, (Fig. 23.) si circulum primum, cujus radius sit R', eum esse ponimus, qui primi anguli (A) crura h. e. latera trianguli b, c tangit etc., esse sequitur:

$$\overset{(1)}{R'} = b.\,\mathrm{tg}\,\tfrac{1}{2}A;\ \overset{(1)}{R''} = c.\,\mathrm{tg}\,\tfrac{1}{2}B;\ \overset{(1)}{R'''} = a.\,\mathrm{tg}\,\tfrac{1}{2}C,$$

eamque ob caussam:

$$R'.\overset{(3)}{R''}.R''' = a\,b\,c.\,\mathrm{tg}\,\tfrac{1}{2}A.\,\mathrm{tg}\,\tfrac{1}{2}B.\,\mathrm{tg}\,\tfrac{1}{2}C.$$

Eadem ratione esse videmus:

$$\overset{(4)}{R'} = c.\,\mathrm{tg}\,\tfrac{1}{2}A;\ \overset{(4)}{R''} = a.\,\mathrm{tg}\,\tfrac{1}{2}B;\ \overset{(4)}{R'''} = b.\,\mathrm{tg}\,\tfrac{1}{2}C,$$

ergo

$$R'.\overset{(4)}{R''}.R''' = a.b.c.\,\mathrm{tg}\,\tfrac{1}{2}A.\,\mathrm{tg}\,\tfrac{1}{2}B.\,\mathrm{tg}\,\tfrac{1}{2}C = R'.\overset{(3)}{R''}.R'''$$

h. e. parallelepipeda recta, a tertiae et quartae ternionis radiis facta, inter se sunt aequalia.

§. 74.

Quum ex praecedetentibus, esse

$$R'.\overset{(1)}{R''}.R''' = \frac{a.b.c}{8\sin A.\sin B.\sin C},$$

cognitum habeamus, vera sit necesse est haec proportio:

$$R'.\overset{(1)}{R''}.R''' : R'.\overset{(3)}{R''}.R''' :: \frac{a\,b\,c}{8\sin A.\sin B.\sin C} : a\,b\,c.\,\mathrm{tg}\,\tfrac{1}{2}A.\,\mathrm{tg}\,\tfrac{1}{2}B.\,\mathrm{tg}\,\tfrac{1}{2}C$$

$$:: \frac{1}{64.\sin\frac{1}{2}A.\sin\frac{1}{2}B.\sin\frac{1}{2}C.\cos\frac{1}{2}A.\cos\frac{1}{2}B.\cos\frac{1}{2}C} : \frac{\sin\frac{1}{2}A.\sin\frac{1}{2}B.\sin\frac{1}{2}C}{\cos\frac{1}{2}A.\cos\frac{1}{2}B.\cos\frac{1}{2}C}$$

$$:: 1 : (8\sin\tfrac{1}{2}A.\sin\tfrac{1}{2}B.\sin\tfrac{1}{2}C)^2$$

In omni autem triangulo est:

$$\sin\tfrac{1}{2}A.\sin\tfrac{1}{2}B.\sin\tfrac{1}{2}C = \frac{r}{4R}\ ^{**});$$

ergo

$$R'.\overset{(1)}{R''}.R''' : R'.\overset{(3)}{R''}.R''' :: 1 : \frac{4r^2}{R^2} :: R^2 : 4r^2,$$

unde

$$R^2 : R'.\overset{(3)}{R''}.R''' :: R^2 : 4r^2$$

**) Est enim: $\sin\frac{1}{2}A = \sqrt{\frac{(a-b+c)(a+b-c)}{4\,b\,c}}$; $\sin\frac{1}{2}B = \sqrt{\frac{(-a+b+c)(a+b-c)}{4\,a\,c}}$

$\sin\frac{1}{2}C = \sqrt{\frac{(-a+b+c)(a-b+c)}{4\,a\,b}}$; ergo

$$\sin\tfrac{1}{2}A.\sin\tfrac{1}{2}B.\sin\tfrac{1}{2}C = \frac{(-a+b+c)(a-b+c)(a+b-c)}{8\,a\,b\,c} = \frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{8(a+b+c)\,a\,b\,c}$$

$$= \frac{2\triangle^2}{(a+b+c).a\,b\,c} = \frac{2\triangle}{a+b+c}.\frac{\triangle}{a\,b\,c} = \frac{r}{4R}$$

$$R : R'.\overset{(3)}{R''}.R''' :: 1 : 4r^2$$

$$\text{vel}\quad R'.\overset{(3)}{R''}.R''' = R'.\overset{(4)}{R''}.R''' = 4R.r^2$$

h. e. parallelepipedum rectum, a tertiae vel quartae ternionis radiis ortum, parallelepipedo aequale est ei, cujus basis diametri minoris quadratum, altitudo autem radium majorem aequat.

§. 75.

Lineas AH, AI, AK, et AH', AI', AK', quarum singula quaevis distantiam centri ab ejus anguli vertice indicat, quem latera, circulum contingentia, intercipiunt, quasque literis δ', δ'', δ''' significabimus, has nobis offerre aequationes apertum est:

$$\overset{(3)}{\delta'} = b.\sec\tfrac{1}{2}A;\quad \overset{(3)}{\delta''} = c.\sec\tfrac{1}{2}B;\quad \overset{(3)}{\delta'''} = a.\sec\tfrac{1}{2}C;\quad \text{et}$$

$$\overset{(4)}{\delta'} = c.\sec\tfrac{1}{2}A;\quad \overset{(4)}{\delta''} = a.\sec\tfrac{1}{2}B;\quad \overset{(4)}{\delta'''} = b.\sec\tfrac{1}{2}C,$$

$$\text{ergo}\qquad \delta'.\overset{(3)}{\delta''}.\delta''' = \delta'.\overset{(4)}{\delta''}.\delta'''$$

h. e. parallelepipeda recta, a lineis, illam, quam modo diximus, distantiam indicantibus, in utraque ternione facta, inter se sunt aequalia.

§. 76.

Quodsi producta, a binis ejusdem ternionis radiis facta, in unam summam colliguntur, haec existat aequatio necesse est:

$$R'.R'' + \overset{(2)}{R'}.R''' + R''.R''' = ab\,\mathrm{tg}\tfrac{1}{2}A.\,\mathrm{tg}\tfrac{1}{2}C + ac\,\mathrm{tg}\tfrac{1}{2}B.\,\mathrm{tg}\tfrac{1}{2}C + bc\,\mathrm{tg}\tfrac{1}{2}A.\,\mathrm{tg}\tfrac{1}{2}B$$

$$\mp ab\sqrt{\frac{(a-b+c)(a+b-c)(-a+b+c)(a-b+c)}{(a+b+c)^2(-a+b+c)(a+b-c)}} + ac\sqrt{\frac{(-a+b+c)(a+b-c)(a+b+c)(a-b+c)}{(a+b+c)^2(a-b+c)(a+b-c)}}$$

$$+ bc\sqrt{\frac{(a+b-c)(a-b+c)(-a+b+c)(a+b-c)}{(a+b+c)^2(a-b+c)(-a+b+c)}} =$$

$$\frac{ab(a-b+c)+ac(-a+b+c)+bc(a+b-c)}{a+b+c} = \frac{a^2b-ab^2+3abc-a^2c+ac^2+b^2c-bc^2}{a+b+c}$$

$$= 6Rr + \left[R' + \overset{(2)}{R''} + R''' - (R' + \overset{(1)}{R''} + R''')\right]\frac{4\triangle}{a+b+c},$$

ergo

$$R'.R'' + \overset{(2)}{R'}.R''' + R''.R''' = \left[3R + R' + \overset{(2)}{R''} + R''' - (R' + \overset{(1)}{R''} + R''')\right]2r$$

h. e. rectangulorum, a binis tertiae ternionis radiis factorum, summa rectangulo aequalis est ei, quod a diametro minori et a differentia inter secundae et primae ternionis radiorum summas, triplo radii majoris aucta, oritur.

§. 78.

Pari modo demonstrari potest, esse:

$$R'.\,R'' + R'.\,\overset{(4)}{R'''} + R''.\,R''' = 2\,r\left[3\,R + R' + \overset{(1)}{R''} + R''' - (R' + \overset{(1)}{R''} + R''')\right].$$

§. 79.

Duarum, quas modo tradidimus, aequationum altera alteri addita, existat necesse est haec:

$$R'.\,R'' + R'.\,\overset{(3)}{R'''} + R''.\,R''' + R'.\,R'' + R'.\,\overset{(4)}{R'''} + R''.\,R''' = 12\,R\,r$$

h. e. rectangulorum, a binis radiis in utraque ternione factorum, summa rectanguli, quod a diametro majori et minori oritur, aream ter complectitur.

§. 80.

Schol. Ne quae sequuntur propositiones, verbis expressae, nimia laborent verborum copia, hunc loquendi usum usurpabimus: rectangula v. c. haec:

$$\overset{(1)}{R'}.\,\overset{(2)}{R''},\quad \overset{(2)}{R'}.\,\overset{(1)}{R'''},\quad \overset{(1)}{R''}.\,\overset{(2)}{R'''}$$

dicemus a primae et secundae ternionis radiis, indicibus permutatis, facta; verum

$$\overset{(2)}{R'}\,\overset{(1)}{R''},\quad \overset{(1)}{R'}\,\overset{(2)}{R'''},\quad \overset{(2)}{R''}\,\overset{(1)}{R'''}$$

rectangulorum a secundae et primae ternionis radiis, indicibus permutatis, factorum nomen gerant etc.; porro si sermo erit de parallelepipedis, a primae secundae et secundae ternionis radiis, indicibus permutatis, factis, haec intelligi volumus:

$$\overset{(1)}{R'}.\,\overset{(2)}{R''}.\,\overset{(2)}{R'''},\quad \overset{(2)}{R'}.\,\overset{(1)}{R''}.\,\overset{(2)}{R'''},\quad \overset{(2)}{R'}.\,\overset{(2)}{R''}.\,\overset{(1)}{R'''};$$

denique rectangula a duabus ternionibus, v. c. prima et secunda, aequabiliter facta, erunt nobis haec:

$$\overset{(1)}{R'}.\,\overset{(2)}{R'},\quad \overset{(1)}{R''}\,\overset{(2)}{R''},\quad \overset{(1)}{R'''}\,\overset{(2)}{R'''}$$

§. 81.

Facile perspicitur esse:

$$\frac{\overset{(3)}{R'}}{\overset{(4)}{R'}} = \frac{b}{c};\quad \frac{\overset{(3)}{R''}}{\overset{(4)}{R''}} = \frac{c}{a};\quad \frac{\overset{(3)}{R'''}}{\overset{(4)}{R'''}} = \frac{a}{b};$$

inde

$$\frac{\overset{(3)}{R'}.\overset{(4)}{R''}.\overset{(4)}{R'''} + \overset{(4)}{R'}.\overset{(3)}{R''}.\overset{(4)}{R'''} + \overset{(4)}{R'}.\overset{(4)}{R''}.\overset{(3)}{R'''}}{R'.\overset{(4)}{R''}.R'''} = \frac{a^2 c + b^2 a + c^2 b}{a\ b\ c};$$

ergo

$$\frac{\overset{(3)}{R'}.\overset{(4)}{R''}.\overset{(4)}{R'''} + \overset{(4)}{R'}.\overset{(3)}{R''}.\overset{(4)}{R'''} + \overset{(4)}{R'}.\overset{(4)}{R''}.\overset{(3)}{R'''}}{4\ R\ r^2} = (R' + \overset{(2)}{R''} + R''')\ \frac{4\ \Delta}{abc}$$

$$= (R' + \overset{(2)}{R''} + R''').\ \frac{1}{R},$$

ergo

$$\overset{(3)}{R'}.\overset{(4)}{R''}.\overset{(4)}{R'''} + \overset{(4)}{R'}.\overset{(3)}{R''}.\overset{(4)}{R'''} + \overset{(4)}{R'}.\overset{(4)}{R''}.\overset{(3)}{R'''} = (R' + \overset{(2)}{R''} + R''')\ 4\, r^2$$

h. e. parallelepipedorum rectorum, a tertiae, quartae et quartae ternionis radiis, indicibus permutatis, factorum, summa parallelepipedo aequalis est ei, cujus basis diametri minoris quadratum, altitudo autem secundae ternionis radiorum summam aequat.

§. 82.

Ratione plane simili, esse

$$\overset{(4)}{R'}.\overset{(3)}{R''}.\overset{(3)}{R'''} + \overset{(3)}{R'}.\overset{(4)}{R''}.\overset{(3)}{R'''} + \overset{(3)}{R'}.\overset{(3)}{R''}.\overset{(4)}{R'''} = (R' + \overset{(1)}{R''} + R''')\ 4\, r^2$$

ostenditur.

§. 83.

Patet omnino esse:

$$\overset{(2)}{R'}.\overset{(4)}{R''} + \overset{(4)}{R'}.\overset{(2)}{R'''} + \overset{(2)}{R''}.\overset{(4)}{R'''} = \frac{a^2.\mathrm{tg}\,\frac{1}{2} B}{2 \sin B} + \frac{b^2.\mathrm{tg}\,\frac{1}{2} C}{2 \sin C} + \frac{c^2.\mathrm{tg}\,\frac{1}{2} A}{2 \sin A}$$

$$= \frac{a^2.\mathrm{tg}\,\frac{1}{2} B}{4 \sin \frac{1}{2} B.\cos \frac{1}{2} B} + \frac{b^2.\mathrm{tg}\,\frac{1}{2} C}{4 \sin \frac{1}{2} C.\cos \frac{1}{2} C} + \frac{c^2.\mathrm{tg}\,\frac{1}{2} A}{4 \sin \frac{1}{2} A.\cos \frac{1}{2} A}$$

$$= \tfrac{1}{4}\left[a^2.\sec^2 \tfrac{1}{2} B + b^2.\sec^2 \tfrac{1}{2} C + c^2.\sec^2 \tfrac{1}{2} A\right]$$

$$= \tfrac{1}{4}\,(\delta'^2 + \overset{(4)}{\delta''^2} + \delta'''^2)$$

h. e. lineae illae, quae distantias inter quartae ternionis centra et eorum angulorum vertices indicant, qui his ex adverso sunt, semper ita sunt comparatae, ut quadratorum, super iis constructorum, summa rectangulorum, a binis secundae et quartae ternionis radiis, indicibus permutatis, factorum, summam quater complectatur.

Cor. Quum sit

$$a^2 \sec^2 \tfrac{1}{2} B + b^2.\sec^2 \tfrac{1}{2} C + c^2.\sec^2 \tfrac{1}{2} A = a^2 + b^2 + c^2 + a^2\,\mathrm{tg}^2 \tfrac{1}{2} B + b^2.\mathrm{tg}^2 \tfrac{1}{2} C + c^2.\mathrm{tg}^2 \tfrac{1}{2} A,$$

recte concludimus esse:

$$\overset{(2)}{R'}.\overset{(4)}{R''}+\overset{(4)}{R'}.\overset{(2)}{R'''}+\overset{(2)}{R''}.\overset{(4)}{R'''}=\tfrac{1}{4}(a^2+b^2+c^2+R'^2+\overset{(4)}{R''^2}+R'''^2),$$

unde

$$\overset{(2)}{R'}.\overset{(4)}{R''}+\overset{(4)}{R'}.\overset{(2)}{R'''}+\overset{(2)}{R''}.\overset{(4)}{R'''}-\tfrac{1}{4}(R'^2+\overset{(4)}{R''^2}+R'''^2)=\tfrac{1}{4}(a^2+b^2+c^2)$$

§. 84.

Pari modo, esse

$$\overset{(3)}{R'}.\overset{(1)}{R''}+\overset{(1)}{R'}.\overset{(3)}{R'''}+\overset{(3)}{R''}.\overset{(1)}{R'''}=\tfrac{1}{4}(\delta'^2+\overset{(3)}{\delta''^2}+\delta'''^2),$$

et

$$\overset{(3)}{R'}.\overset{(1)}{R''}+\overset{(1)}{R'}.\overset{(3)}{R'''}+\overset{(3)}{R''}.\overset{(1)}{R'''}-\tfrac{1}{4}(R'^2+\overset{(3)}{R''^2}+R'''^2)=\tfrac{1}{4}(a^2+b^2+c^2),$$

demonstrari potest.

Cor. Sit igitur necesse est:

$$\overset{(3)}{R'}.\overset{(1)}{R''}+\overset{(1)}{R'}.\overset{(3)}{R'''}+\overset{(3)}{R''}.\overset{(1)}{R'''}-\tfrac{1}{4}(R'^2+\overset{(3)}{R''^2}+R'''^2)=\overset{(2)}{R'}.\overset{(4)}{R''}+\overset{(4)}{R'}.\overset{(2)}{R'''}+\overset{(2)}{R''}.\overset{(4)}{R'''}-\tfrac{1}{4}(R'^2+\overset{(4)}{R''^2}+R'''^2)$$

§. 85.

Non minus apertum est, esse omnino:

$$\overset{(4)}{R'}.\overset{(2)}{R''}+\overset{(2)}{R'}.\overset{(4)}{R'''}+\overset{(4)}{R''}.\overset{(2)}{R'''}=\frac{b.c.\operatorname{tg}\frac{1}{2}A}{2\sin C}+\frac{a.b.\operatorname{tg}\frac{1}{2}C}{2\sin B}+\frac{a.c.\operatorname{tg}\frac{1}{2}B}{2\sin A}$$
$$=R\,[b.\operatorname{tg}\tfrac{1}{2}A+a.\operatorname{tg}\tfrac{1}{2}C+c.\operatorname{tg}\tfrac{1}{2}B]$$
$$=R\,[R'+\overset{(3)}{R''}+R''']$$

h. e. rectangulorum, a quartae et secundae ternionis radiis, indicibus permutatis, factorum, summa rectangulo aequatur ei, quod a radio majori et tertiae ternionis radiorum summa ortum est.

§. 86.

Eadem fere via nanciscimur hanc aequationem:

$$\overset{(1)}{R'}.\overset{(3)}{R''}+\overset{(3)}{R'}.\overset{(1)}{R'''}+\overset{(1)}{R''}.\overset{(3)}{R'''}=R.(R'+\overset{(4)}{R''}+R''')$$

§. 87.

Ex iis, quae modo tradidimus, esse sequitur:

$$\frac{\overset{(4)}{R'}.\overset{(2)}{R''}+\overset{(2)}{R'}.\overset{(4)}{R'''}+\overset{(4)}{R''}.\overset{(2)}{R'''}}{R'+\overset{(3)}{R''}+R'''}=R=\frac{\overset{(1)}{R'}.\overset{(3)}{R''}+\overset{(3)}{R'}.\overset{(1)}{R'''}+\overset{(1)}{R''}.\overset{(3)}{R'''}}{R'+\overset{(4)}{R''}+R'''},$$

unde

$$(\overset{(4)}{R'}.\overset{(2)}{R''}+\overset{(2)}{R'}.\overset{(4)}{R'''}+\overset{(4)}{R''}.\overset{(2)}{R'''}).(R'+\overset{(4)}{R''}+R''')=(\overset{(1)}{R'}.\overset{(3)}{R''}+\overset{(3)}{R'}.\overset{(1)}{R'''}+\overset{(1)}{R''}.\overset{(3)}{R'''}).$$
$$(R'+\overset{(3)}{R''}+R''')$$

h. e, parallelepipedum, cujus basis rectangulorum, a quartae et secundae ternionis radiis, indicibus permutatis, factorum, aream, altitudo autem quartae ternionis radiorum summam complectitur, parallelepipedo aequatur ei, cujus basis rectangulorum, a prima et tertia ternione, permutatis indicibus, ortorum, summam, altitudo vero tertiae ternionis radiorum summam aequat.

§. 88.

Nemo non intelligit, esse:

$$\overset{(2)}{R'}.\overset{(3)}{R''}=\frac{a.\ c.\ \mathrm{tg}\ \tfrac{1}{2}\ B}{2\sin B}=\frac{a.\ c.\ \sec^2\ \tfrac{1}{2}\ B}{4}$$

et

$$\overset{(4)}{R'}.\overset{(1)}{R''}=\frac{a.\ c.\ \mathrm{tg}\ \tfrac{1}{2}\ B}{2\sin B}=\frac{a.\ c.\ \sec^2\ \tfrac{1}{2}\ B}{4}$$

ergo $\overset{(2)}{R'}.\overset{(3)}{R''}=\overset{(4)}{R''}.\overset{(1)}{R'''}$, similique ratione

$$\overset{(3)}{R'}.\overset{(2)}{R'''}=\overset{(4)}{R'}.\overset{(1)}{R''}$$
$$\overset{(2)}{R''}.\overset{(3)}{R'''}=\overset{(1)}{R'}.\overset{(4)}{R'''}$$

h. e. rectangulum, ab iis secundae tertiaeque ternionis radiis factum, quorum circuli ejusdem lateris fines tangunt, rectangulo aequatur ei, quod ab iis primae et quartae ternionis radiis ortum est, quorum circuli posterioris lateris fines contingunt.

§. 89.

Quum esse pateat:

$$ab\sec^2\tfrac{1}{2}C+ac\sec^2\tfrac{1}{2}B+bc\sec^2\tfrac{1}{2}A=ab(1+\mathrm{tg}^2\tfrac{1}{2}C)+ac(1+\mathrm{tg}^2\tfrac{1}{2}B)+bc(1+\mathrm{tg}^2\tfrac{1}{2}A)$$
$$=ab+ac+bc+ab\,\mathrm{tg}^2\tfrac{1}{2}C+ac\,\mathrm{tg}^2\tfrac{1}{2}B+bc\,\mathrm{tg}^2\tfrac{1}{2}A,$$

hinc esse colligimus:

$$\overset{(2)}{R'}.\overset{(3)}{R''}+\overset{(3)}{R'}.\overset{(2)}{R'''}+\overset{(2)}{R''}.\overset{(3)}{R'''}=\overset{(1)}{R'}.\overset{(4)}{R''}+\overset{(1)}{R'}.\overset{(4)}{R'''}+\overset{(4)}{R''}.\overset{(1)}{R'''}=\overset{(3)}{R'}.\overset{(2)}{\mathrm{R}'}+\overset{(1)}{\mathrm{R}''}.\overset{(3)}{\mathrm{R}''}+$$
$$+\overset{(3)}{\mathrm{R}'''}.\overset{(1)}{\mathrm{R}'''}+r.r'+r.r''+r.r'''+r'.r''+r'.r'''+r''.r'''$$

h. e. rectangulorum, a secunda et tertia ternione, indicibus permutatis, factorum, summa rectangulorum, a quarta et prima ternione eodem modo ortorum, summae aequatur;

utraque autem summa rectangulorum a binis radiis, quorum circuli latera trianguli contingunt, effectorum summam, rectangulis a tertia et quarta ternione aequabiliter factis auctam, aequat.

§. 90.

Praeterea tertia et quarta ternio hanc quoque nobis praebent aequationem:

$$\overset{(3)}{R'}.\overset{(4)}{R''} + \overset{(4)}{R'}.\overset{(3)}{R'''} + \overset{(3)}{R''}.\overset{(4)}{R'''} = ab \operatorname{tg} \tfrac{1}{2} A.\operatorname{tg} \tfrac{1}{2} B + ac \operatorname{tg} \tfrac{1}{2} A.\operatorname{tg} \tfrac{1}{2} C + bc \operatorname{tg} \tfrac{1}{2} B.\operatorname{tg} \tfrac{1}{2} C$$

quae eodem, quo supra (§. 76.) factum est, modo transformata abit in hanc:

$$\overset{(3)}{R'}.\overset{(4)}{R''} + \overset{(4)}{R'}.\overset{(3)}{R'''} + \overset{(3)}{R''}.\overset{(4)}{R''} = \frac{ab\,(a+b-c) + ac\,(a-b+c) + bc\,(-a+b+c)}{a+b+c}$$

$$= \frac{a^2 b + b^2 c + c^2 a + a^2 c + b^2 a + c^2 b - 3\,abc}{a+b+c}$$

$$= \frac{4\,\triangle\,(R' + \overset{(1)}{R''} + R''' + R' + \overset{(2)}{R''} + R''') - 3\,abc}{a+b+c}$$

$$= \frac{4\,\triangle}{a+b+c}\,(R' + \overset{(1)}{R''} + R''' + R' + \overset{(2)}{R''} + R''' - 3R)$$

$$= 2r\,(R' + \overset{(1)}{R''} + R''' + R' + \overset{(2)}{R''} + R''' - 3\,R)$$

h. e. rectangulorum, a tertia et quarta ternione, indicibus permutatis, factorum, summa illi rectangulo aequatur, quod a diametro minori et linea ortum est ea, qua primae et secundae ternionis radiorum summa radii majoris triplum excedit.

§. 91.

In iisdem ternionibus hanc aequationem locum habere patet:

$$\overset{(4)}{R'}.\overset{(3)}{R''} + \overset{(3)}{R'}.\overset{(4)}{R'''} + \overset{(4)}{R''}.\overset{(3)}{R'''} = a^2.\operatorname{tg} \tfrac{1}{2} B.\operatorname{tg} \tfrac{1}{2} C + b^2.\operatorname{tg} \tfrac{1}{2} A.\operatorname{tg} \tfrac{1}{2} C + c^2.\operatorname{tg} \tfrac{1}{2} A.\operatorname{tg} \tfrac{1}{2} B$$

$$= \frac{a^2(-a+b+c) + b^2(a-b+c) + c^2(a+b-c)}{a+b+c}$$

$$= \frac{a^2 b + b^2 c + c^2 a + a^2 c + b^2 a + c^2 b - (a^3 + b^3 + c^3)}{a+b+c}$$

$$= \frac{4\,\triangle}{a+b+c}\left[R' + \overset{(1)}{R''} + R''' + R' + \overset{(2)}{R''} + R''' - \frac{\overset{(1)}{R'}.\overset{(2)}{R'} + \overset{(1)}{R''}.\overset{(2)}{R''} + \overset{(1)}{R'''}.\overset{(2)}{R'''}}{R}\right],$$

ergo

$$R[\overset{(4)}{R'}.\overset{(3)}{R''}+\overset{(3)}{R'}.\overset{(4)}{R'''}+\overset{(4)}{R''}.\overset{(3)}{R'''}]=2r[R'.R''+\overset{(1)}{R'}.R'''+R''.R'''+R'R''+\overset{(2)}{R'}.R'''+R''R'''-(R'R'+\overset{(1)}{R''}\overset{(2)}{R''}+R'''.R''')]$$

h. e. parallelepipedum rectum, a summa rectangulorum, quae a quartae et tertiae ternionis radiis, indicibus permutatis, orta sunt, et radio majori factum, parallelepipedo aequale est ei, cujus basis aream, qua rectangulorum, a binis radiis et primae, et secundae ternionis, effectorum summa, summam eorum, quae ab iisdem ternionibus aequabiliter facta sunt, excedit, altitudo autem diametrum minorem aequat.

Sectio commentationis quarta.

Syntheticae theorematum aliquot, plerumque in praecedentibus jam tractatorum, demonstrationes.

§. 92.

Theor. Quodsi, super trianguli cujusdam lateribus triangulis (Fig. 24.) aequilateris *ABK*, *ACI*, *BCH*, constructis, ducuntur lineae *AH*, *BI*, *CK*, anguli *AGK*, *BGK*, *BGH* inter se sint aequales necesse est.

Dem. Ejusmodi lineas in eodem puncto sese invicem transire ex iis, quae supra (§. 39.) leguntur, facile sequitur. Angulos autem, ad quos secantur, invicem aequari, hoc modo probamus. Quum triangula *ABH* et *BCK* sibi congruant, sit $< BKG = < BAC$ necesse est; ergo quadrangulo *AGBK* circulus potest circumscribi, eamque ob caussam est

$$< BGK = < BAK = \tfrac{2}{3} R.$$

Ejusdem magnitudinis ceteros esse angulos, eodem fere modo demonstratur.

Schol. 1. Nil plane refert, ab utra primarii laterum parte puncta *H*, *I*, *K* sita sint, modo ne desideretur hujus situs aequabilitas.

Schol. 2. Theorematis nostri ope facili negotio hoc problema solvitur: in quolibet triangulo transversas eas ducere, quae uno puncto ita secantur, ut omnes anguli, ab iis constituti, inter se sint aequales.

§. 93.

Theor. In utroque fine cujusvis laterum, triangulum quodlibet constituentium, perpendiculis AI, AK, BH, BK, CH, CI (Fig. 25.) erectis, rectae, AH, BI, CK in ipso circuli exterioris centro invicem secentur necesse est.

Dem. Primum enim, quum ex constructione sit $< KAB = < CAI$ etc., has lineas unum punctum transire constat (§. 39.); deinde vero quadrangula $ACBK$ et $ABCI$ ita sunt comparata, ut circuli iis circumscribi possint; ergo

$$< BCK = < BAK = < CAI = < CBI;$$

unde

$$\triangle\ BCK \cong \triangle\ BCI;$$

ergo $BCIK$ est rectangulum, adeoque

$BG = CG$, similique caussa

$BG = GA$;

ergo G centrum esse circuli exterioris videmus.

Cor. 1. Circulum illum, qui triangulo primario circumscriptus est, simul et puncta H, I, K transire, facile intelligitur.

Cor. 2. Triangula LMN et OPQ (Fig. 31.), primario ita circumscripta, ut ejus lateribus ad angulos rectos occurrant, invicem sibi congruunt. Est enim: $KL = CO$, $BN = IP$, $BK = CI$, ergo $LN = PO$ adeoque triangula ipsa sibi congrua.

§. 94.

Theor. Quodsi, dato quolibet triangulo ABC (Fig. 26.), tres circuli $BCMI$, $ACNK$, $ABLH$ ita describuntur, ut primus latus AC, secundus AB, tertius denique latus BC contingat, ductis quibuspiam lineis transversis LAN, et CNM, puncta M, B, L omnino ita esse comparata, ut a primo per secundum ad tertium recta duci possit, contendimus.

Dem. Ducantur quidem rectae BM, LB; est autem

$$< ANC = < AGC = < ABC + < ACB$$

$$\text{ergo} \quad < LNM = < BAC;$$

porro esse videmus:

$$< BMC = < EGC = < ACB$$

et

$$< BLA = < BGD = < ABC$$

quare sit

$$< ABL + < ABM = 2R$$

necesseest; itaque vera esse, quae diximus, adparet.

Cor. Triangulum igitur LMN, quod primario simile esse videmus, et hanc ob caussam ad illa referendum, quae supra (§. 44.) tractavimus, ita comparatum est, ut ejus vertices in circulorum nostrorum siti sint circumferentiis. Idem valere de omnibus triangulis, huic classi adnumerandis, facile intelligitur.

§. 95.

Theorematis illius elegantissimi, jam supra (§. 43.) a nobis traditi, nunc alia, eaque simplicissima, confici potest demonstratio. Nam anguli BLG et BAG, tanquam ad extremitatem circuli eidem arcui innitentes, sint inter se aequales necesse est; eâdem vero caussa esse:

$$< GMC = < GBC$$
$$\text{et} \quad < GNA = < GCA$$

facile perspicitur.

§. 96.

Theor. Iisdem circulis descriptis, quae transversae AK, BH, CI ita ducuntur, ut sit

$$< BAK = < CBH = < ACI = 2. < BAG,$$

hae triangulum HIK primario congruum efficiant necesse est.

Dem. Quum enim ex constructione ipsa sit $< GCK = < GCA = < GIK$, et hanc ob caussam $CG = GI$, triangula AGC et GIK, quae inter se similia esse, facili negotto intelligitur, sibi invicem congruere apparet; ergo $AC = IK$, adeoque $\triangle ABC \cong \triangle HIK$.

§. 97.

Theor. Omnium triangulorum, primario similium, quae a transversis, eo, quo antea factum est, modo ductis, efficiuntur, maximum est illud HIK (Fig. 27.), cujus latera transversas AD, BE, CF ad angulos rectos secant.

Dem. Ipsius hujus trianguli aream esse majorem area cujusvis alius trianguli, ad hanc classem referendi, v. c. OPQ probari potest. Quum enim lineae GI, GO ita comparatae sint, ut in triangulis similibus puncta, quibus est situs aequabilis, jungant, eandem earum rationem, quae est binorum in his triangulis laterum sibi respondentium, esse constat; atqui IG, quoniam angulum GCI 90 gradus explere patet, circuli $BGCI$ est diametrus, eamque ob caussam maxima omnium, quae in illo duci possunt, chordarum; ergo quodvis trianguli HIK latus latere, in triangulo OPQ ei respondente, est majus, adeoque HIK omnium ejusmodi triangulorum maximum.

§. 98.

Theor. Idem illud triangulum *HIK*, quod modo memoravimus, maximum trianguli *OPQ*, cujus latera primarii lateribus ad angulos rectos occurrunt, et primarii areas complectitur.

Dem. Quum enim in tribus his triangulis lineae *IG*, *OG*, *CG* sibi respondeant, nunc quoque earum ratio laterum ipsorum, sibi respondentium, rationem aequat. Est autem ex constructione *CO* circuli *RGCI* diametrus, et hanc ob caussam $= GI$; ergo, sit

$$\overline{GI}^2 = \overline{CO}^2 = \overline{GO}^2 + \overline{CG}^2$$

necesse est; quare, quum triangula *HIK*, *OPQ*, *ABC* inter se sint similia, esse

$$\triangle\ HIK = \triangle\ OPQ + \triangle\ ABC,$$

concludamus licet.

§. 99.

Theor. Quodsi in triangulo quopiam *ABC* transversae *AL*, *BM*, *CN* ita ducuntur, ut sit (Fig. 28.)

$$< ALC = < BMA = < CNB = \tfrac{2}{3}\ R$$

triangulum *HIK*, ab iis factum, primario dicimus esse congruum.

Dem. Cognovimus enim triangulum *HIK* primario esse simile punctumque *G*, quo perpendicula, ab hujus verticibus in latera demissa, invicem secantur, circuli, illi circumscribendi, centrum; jam, si est *O* centrum circuli, qui primarii vertices transit, demissis *OP* et *GR*, esse

$$AG = 2\ GR$$

necesse est, quum sit ex constructione ipsa

$$< GAR = \tfrac{1}{3}\ R;$$

non minus autem, esse

$$AG = 2\ OP$$

constat; ergo

$$OP = GR$$

eamque ob caussam triangula *BOP* et *GRK* similia sibi congruant necesse est; ergo

$$BO = GK$$

adeoque ipsa triangula *ABC*, *HIK*, quae non solum inter se similia, sed etiam sunt circulis aequatibus inscripta, sibi congruere patet.

§. 100.

Theor. Quodsi, dato quopiam triangulo ABC (Fig. 29.), tres circuli, exteriori ejus circulo aequales, ita describuntur, ut non nisi binos trianguli vertices transeant, eodem illo, quo perpendicula, a verticibus ad latera demissa, invicem secentur puncto necesse est.

Dem. Ex constructione ipsa,

$$< BLC = A$$

esse videmus; ergo, quum

$$< BGC = B + C$$

esse constet, fieri non potest, quin sit

$$< BLC + < BGC = 2\ R,$$

adeoque circulus BLC punctum G transeat. Eodem fere modo ceteros duos circulos idem illud punctum transire, probatur.

§. 101.

Theor. Omnia illa triangula, primario similia, quae a transversis, tertiae ternionis lineas ad angulos aequales secantibus, efficiuntur, ita comparata esse dicimus, ut eorum vertices in circulorum nostrorum siti sint circumferentiis.

Dem. Facili negotio perspicitur, quadrangula $AGBN$, $BGCL$, $AGCM$ (Fig. 29.) ad ea esse referenda, quibus circuli possint circumscribi, ex quo vera esse, quae diximus, elucet.

§. 102.

Theor. Quorum quidem triangulorum maximum est illud LMN (Fig. 29.), cujus latera tertiae ternionis lineas ad angulos rectos secant.

Dem. Circulorum, qui ejusmodi triangulis circumscribi possunt, radios esse chordas GH, GN etc., quum cognitum habeamus, lineam autem GN diametrum esse pateat, circulus is, qui triangulo LMN circumscribitur, omnium sit maximus, adeoque ipsius trianguli area maxima necesse est.

Cor. Maximum hoc triangulum primarii aream quater complecti, per se satis adparet.

§. 103.

Theor. Quod triangulum HIK fit a transversis, tertiae ternionis lineas ad angulos 30 graduum secantibus, primario sit congruum necesse est.

Dem. Ex ipsa constructione esse patet

$$< GAH = < GNA = \tfrac{1}{3}\ R$$

ergo $GH = \frac{1}{2}\ GN$; quare triangula ipsa, ABC et HIK, quum similia, et circulis ejusdem magnitudinis sint inscripta, sibi invicem congruere adparet.

Schol. Haec theorematis nostri elegantissimi demonstratio illa, quam supra (§. 98.) tradidimus, simplicior est atque expeditior.

§. 104.

Theor. Quae transversae ita ducuntur, ut tertiae ternionis lineis ad angulos 45 graduum occurrant, triangulum, primarii duplum, efficiunt.

Dem. Tunc enim triangulum GHN non solum est rectangulum, sed etiam aequicrurum; ergo

$$\overline{GN}^2 = 2\,\overline{GH}^2$$

et hanc ob caussam

$$\triangle HIK = \tfrac{1}{2} \triangle LMN = 2 \triangle ABC$$

Schol. Eodem fere modo, triangulum illud, cujus latera tertiae ternionis lineas ad angulos 60 graduum secant, primarii aream ter complecti, probatur; est enim nunc $\overline{GH}^2 = \tfrac{4}{3}\,\overline{GN}^2$ etc.

§. 105.

Theor. Quodsi transversae AD, BE, CF, ita ducuntur, ut triangulum quoddam HIK (Fig. 30.), constituant, circulos AIB, BKC, CHA in uno puncto sese invicem transire necesse est.

Dem. Binos enim circulos nostros invicem secari per se satis apparet; jam vero si sit G punctum id, quod circuli AIB et CHA una transeunt, fieri non potest, quin circulus tertius BKC idem punctum transeat. Nam ex constructione ipsa

$$< GBI = < GAI = < GAH = < GCH = < GCK$$

esse sequitur; ergo quadrangulo $BKGC$ circulus potest circumscribi, vel quod idem est, qui circulus puncta B, C, K transit, idem punctum G transeat necesse est.

Cor. 1. Angulos, ad quos transversae AGL, BGM, CGN invicem secantur angulis trianguli HIK, si singulum singulo compares, aequales esse facile intelligitur.

Cor. 2. Theorematis nostri ope haec duo problemata facile solvi possunt:

1) Dato quolibet triangulo, transversas eas ducere, quae uno puncto invicem ita secentur, ut angulorum, ad quos sese transeunt, cuique singulo certa sit magnitudo.

2) Dato quolibet triangulo transversas ducere, a quibus triangulum, quod dato alteri sit simile, efficiatur, vel quod idem valet: triangulum construere, quod dato triangulo sit simile cujusque latera vel ipsa vel producta, puncta tria data transeant.

§. 106.

Lemma. Quae perpendicula DH, EI, FK in trianguli ABC lateribus ita eriguntur, ut sit

$$AF.\,AB + BD.\,BC + CE.\,CA = AE.\,AC + BF.\,BA + CD.\,CB,$$

haec in uno puncto invicem sese transire dicimus.

Dem. Si neges, sit LM perpendiculum id, quod idem punctum transit, quo EI, FK secantur; tunc esse

$$BM.\,BC + CE.\,CA + AF.\,AB = CM.\,CB + AE.\,AC + BF.\,BA \text{ (§. 22.)}$$

necesse esset; ex quo, si cum hypothesi compares, concludas licet, esse

$$BM\,BC - BD.\,BC = CM.\,CB - CD.\,CB$$

$$BM. - BD = CM - CD$$

$$- MD = + MD$$

quod quidem absurdum, nisi $MD = o$ h. e. nisi HD est perpendiculum illud, quod idem punctum, quo EF, FK secantur, transit.

§. 107.

Theor. Quodsi tres circuli $ATPW$, $ALCV$, et $BNCU$ (Fig. 33) ita describuntur, ut binos trianguli ABC vertices transeant, lineas AXD, BYE, CZF, puncta, quibus bini circuli invicem secantur, jungentes, in eodem puncto sese invicem transire necesse est.

Dem. Brevitatis caussa, esse

$$CH = R'; \; CI = R'', \text{ et } AK = R''',$$

ponamus; ductis autem lineis HI, HK, IK, per se satis apparet, esse:

$$R'^2 - \overline{SY}^2 + R''^2 - \overline{ZM}^2 + R'''^2 - \overline{PX}^2 = R'^2 - \overline{ZM}^2 + R''^2 - \overline{PX}^2 + R'''^2 - \overline{SY}^2,$$

inde

$$(R' - RS)^2 + (R'' - LM)^2 + (R''' - PO)^2 = (R' - MN)^2 + (R'' - PQ)^2 + (R''' - ST)^2;$$

ergo

$$(R' - RS)^2 + (R' - RS)\,(R''' - ST) + (R'' - LM)^2 + (R'' - LM)\,(R' - MN) + (R''' - PO)^2 + (R''' - PO)(R'' - PQ) = (R' - MN)^2 + (R' - MN)\,(R'' - ML) + (R'' - PQ)^2 + (R'' - PQ)\,(R''' - PO) + (R''' - ST)^2 + (R''' - ST).(R' - RS),$$

unde

$$(R' - RS)\,(R' - RS + R''' - ST) + (R'' - LM)\,(R'' - LM + R' - MN) + (R''' - PO)\,(R''' - PO + R'' - PQ) = (R' - MN)\,(R' - MN + R'' - ML) + (R'' - PQ)\,(R'' - PQ + R''' - PO) + (R''' - ST)\,(R''' - ST + R' - RS),$$

ergo

$$HS.\,HK + IM.\,IH + PK.\,IK = HM.\,HI + IP.\,KI + SK.\,HK$$

eamque ob caussam lineae AD, BE, CF in eodem puncto sese transeant necesse est (§. 106.).

Plura quidem superesse, quae a nobis tractari potuissent, vel potius, ut justae argumenti tractationi nil deesset, tractanda fuissent, quamquam nos non fugit, tamen, ne fines hujus commentationis transgrediamur, hic nobis est subsistendum veniendumque ad ea, quae illius conscribendae opportunitatem nobis adtulerunt. Instat nimirum dies ille faustissimus et auspicatissimus, qui scholae nostrae habetur natalis. Ad hunc pie riteque celebrandum hora VIII. matut. magistri ac discipuli festa pompa templum adibunt, pio gratoque animo tot ac tanta beneficia, quibus Deus O. M. per annum proxime elapsum scholam cumulavit, recordaturi, summamque ejus benignitatem imploraturi, qua et in posterum Portam nostram in patriae salutem vigere ac florere velit. Iisdem deinde hora IX. in Auditorio Majori congregatis, discipuli, ex omnibus classibus electi, ad orationes et recitationes habendas prodibunt, ita quidem, ut qui sunt inferiorum classium aliena, qui vero duarum superiorum pronuntient sua et hoc ordine:

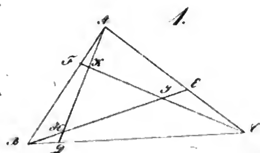
Ex classe secunda superiori *Adolphus Lange* Portensis, *Bertholdus Knauth* Lipsiensis, *Adolphus Hüllmann* Mariaeburgensis, et *Agathon Wunderlich* Gottingensis carmina vernaculo sermone a se composita recitabunt; tum *Julius Seeburg* Sorbigensis oratione latina Portae ipsum nomen gratulabitur.

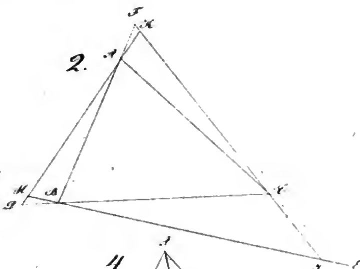
Ex classe prima *Henricus Buchrucker* Sanderslebiensis oratione vernacula hoc argumentum tractabit. „Ursachen der Ausartung der deutschen Sprache vom Ende des dreizehnten Jahrhunderts.“ *Ludovicus Hornemann* Gubenensis carmine latino cantabit: Hussum vatem. Carolus *Kitzing Brehnensis* oratione latina ostendere studebit: „Dignitati et praestantiae religionis Christianae operibus artium sculpturae et picturae, quae aedes, conventibus sacris destinatas, ornant, nihil detrahi.“

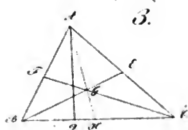
His peractis Rector scholae discipulos eos evocabit, qui ex magistrorum sententia cum strenuo literarum studio, tum morum candore ac probitate sunt insignes, et praemiis studii frugiferi inter eos distribuendis finem imponet actui, cui ut adesse velint Curatores scholae Illustrissimi et omnes, qui Portae ejusque rebus favent, omni, qua par est, observantia rogamus.

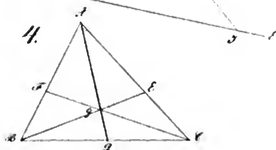
Errata.

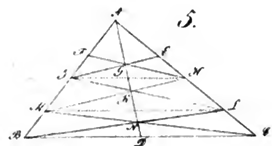
pag. 5 lin 19 pro $g = h$ etc. legatur $k = h$ etc.
— 25 — 12 pro $2\, a' b' c' + 2 = \ldots + a'' b'' c''$ legatur $2\, a' b' c' = \ldots + 2\, a'' b'' c''$
— 29 — 3 et 5 pro $\frac{\sin C}{a\, b}$ legatur $\frac{a\, b}{\sin C}$
— — — 6 — $\frac{\sin A}{b\, c}$ — $\frac{b\, c}{\sin A}$
— — — 10 — $\frac{a\, c}{\sin B}\ \frac{b\, c}{\sin A}$ legatur $\frac{a\, c}{\sin B} + \frac{b\, c}{\sin A}$

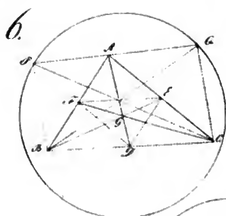

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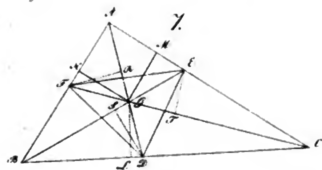

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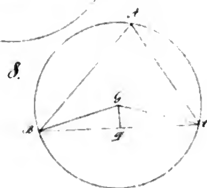

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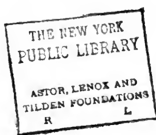

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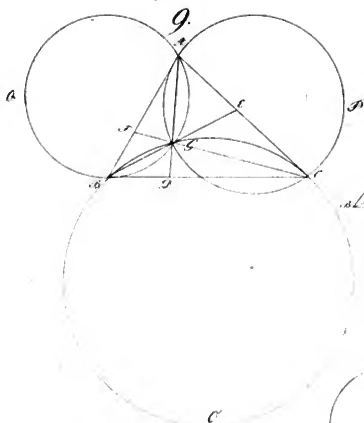

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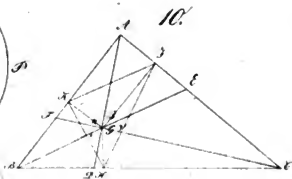

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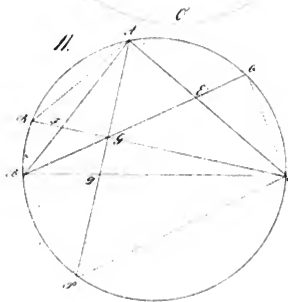

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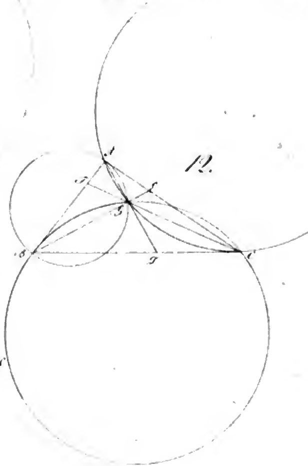

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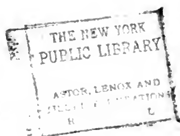


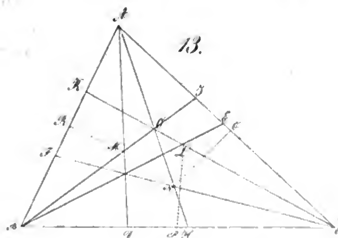

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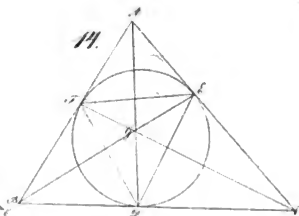

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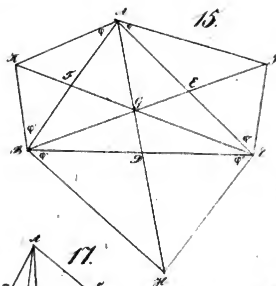

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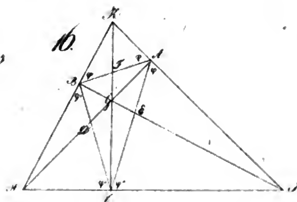

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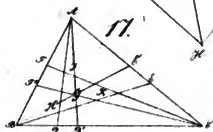


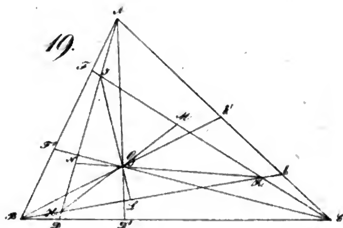

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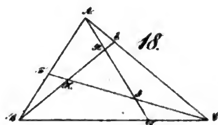

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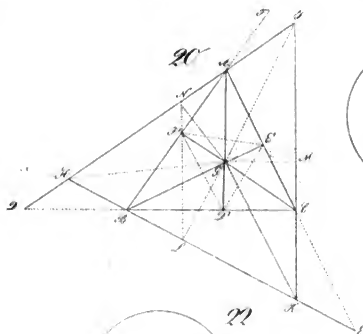

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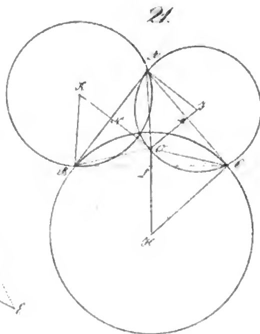

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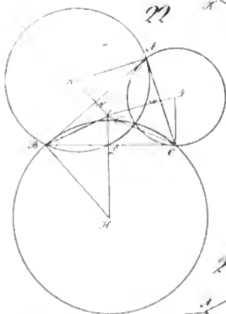

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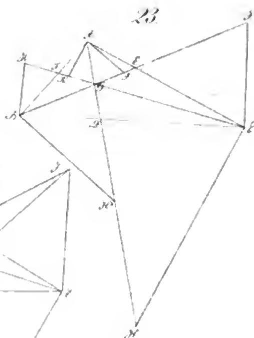

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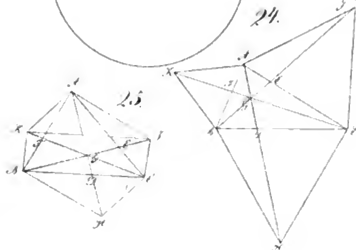

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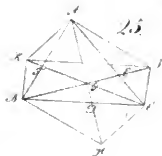

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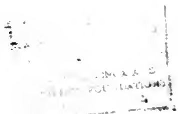

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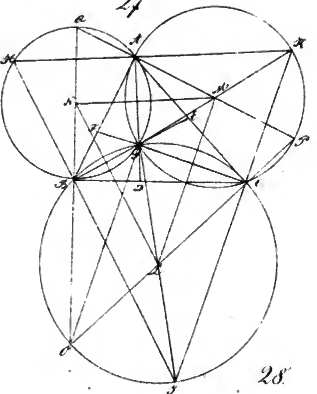

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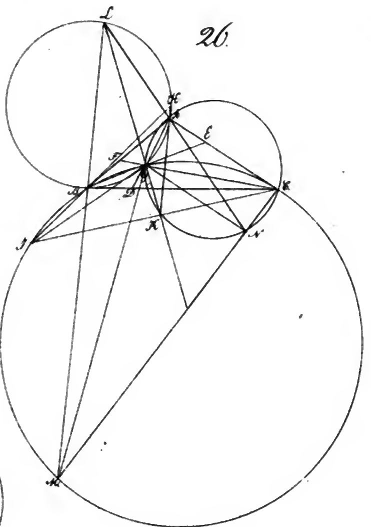

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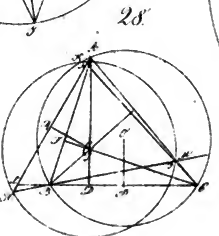

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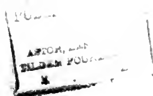

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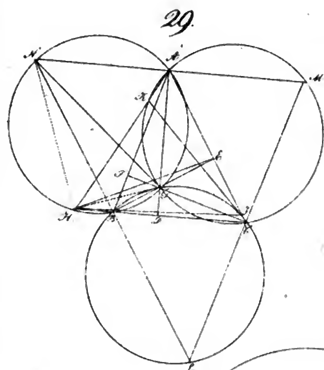


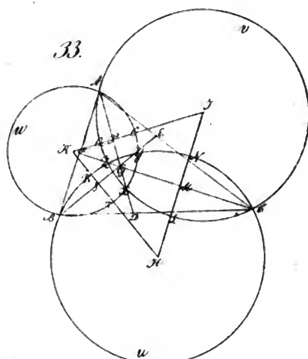

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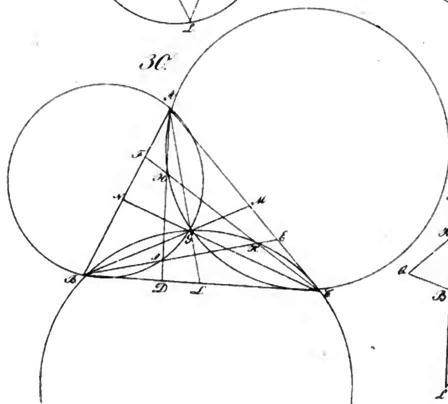

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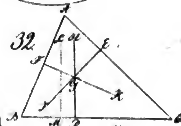

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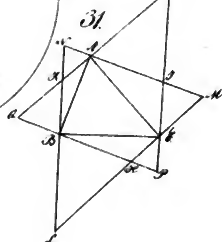



29.


33.


30.


32.


31.